CHAPTER 8

TRANSFORMATION GEOMETRY

8.1 TRANSFORMATIONS AND ISOMETRIES

KEY IDEAS

A geometric **transformation** changes the position, size, or shape of a figure. A particular type of transformation is defined by describing where it "takes" each point of a figure. Under a transformation, each point of the original figure is taken to its **image**. The original point is called the **preimage**. An **isometry** is a transformation that preserves the distance between points. If two geometric figures can be related by an isometry, then they are congruent.

Transformation Notation

Figure 8.1 shows one possible transformation of $\triangle ABC$ onto $\triangle A'B'C'$. The set of points that comprise $\triangle ABC$ belong to the *preimage* set and the set of points that form $\triangle A'B'C'$ represent the *image* set.

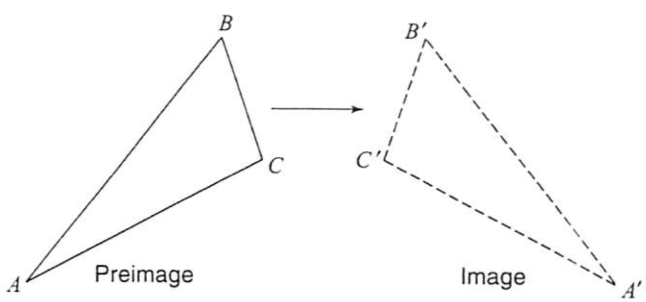


Figure 8.1 Transformation of $\triangle ABC$ onto $\triangle A'B'C'$.

It is customary to use the same capital letter to represent a preimage point and its matching image point. To distinguish between the two points, the letter representing the image point is followed by a prime mark ($'$), as when A' is the image of A. The pairing of a preimage image point with its corresponding image point can also be indicated using "arrow" notation. Referring to Figure 8.1, $A \to A'$ indicates that point A' is the image of point A under the given transformation. Similarly, the notation $\triangle ABC \to \triangle A'B'C'$ means that $\triangle A'B'C'$ is the image of $\triangle ABC$ under the given transformation.

Definition of Transformation

Because each point of $\triangle ABC$ in Figure 8.1 corresponds to exactly one point of $\triangle A'B'C'$ and each point of $\triangle A'B'C'$ corresponds to exactly one point of $\triangle ABC$, there exists a one-to-one mapping or pairing of the points of the two figures.

MATH FACTS

A **transformation** is a one–to–one mapping of the points of one set, called the **preimage** set, onto the points of a second set, called the **image** set. When a transformation is applied to a geometric figure, the transformed figure is the image and the original figure is the preimage.

Congruence Transformations

A transformation that maintains the distance between any two points of a figure is called an **isometry**. Under any isometry:

- Collinearity and betweenness of points are preserved. In Figure 8.2, A' and B' are the image points of A and B under some isometry. If C is between A and B, then C' is between A' and B'.
- The image of a line segment is a congruent line segment. In Figure 8.2, if $AB = 6$, then $A'B' = 6$.
- The image of an angle is a congruent angle, as illustrated in Figure 8.2.

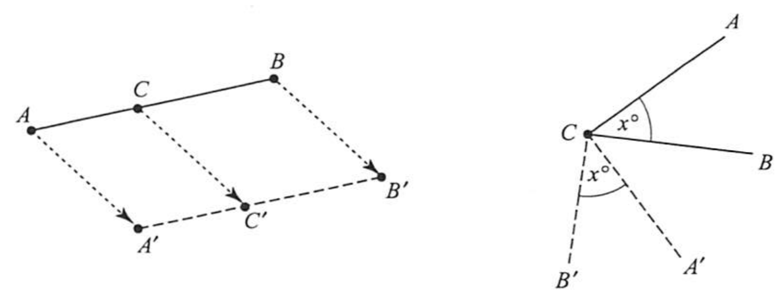


Figure 8.2 Properties of Isometries.

Some Basic Isometries

Figure 8.3 illustrates three different types of transformations that are isometries: reflection, translation, and rotation. Because an isometry always produces an image congruent to the original figure, it is sometimes referred to as a **congruence transformation**.

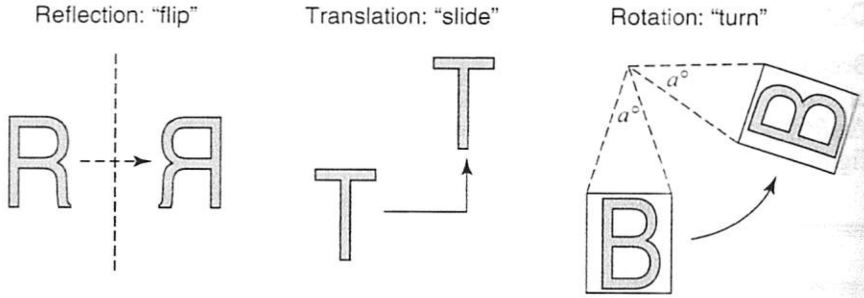


Figure 8.3 Transformations that are isometries.

Reflections

A line *reflection* "flips" an object over the line so that the image appears "backwards" much like how the reflected image of that object would appear in a mirror. The transformation represented in Figure 8.1 is actually the reflection of $\triangle ABC$ over a vertical line (not drawn) midway between $\triangle ABC$ and $\triangle A'B'C'$.

Figure 8.4 shows how to determine the reflected image of point A over line ℓ:

- Draw a line segment from A perpendicular to line ℓ. Extend that segment its own length to A'.
- The shorthand notation $r_\ell(A) = A'$ indicates the reflection of point A over line ℓ is A'.

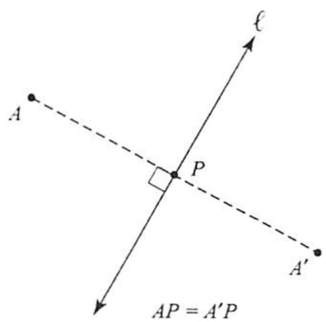


Figure 8.4 Reflecting point A over line ℓ.

To reflect a polygon over a line, reflect each of its vertices over the line. Then connect the reflected image points with line segments as shown in Figure 8.5.

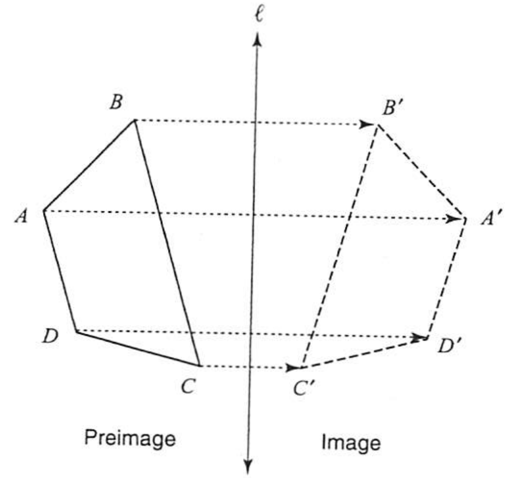


Figure 8.5 Reflecting Trapezoid *ABCD* over line ℓ.

MATH FACTS

A **reflection** over a line is an isometry that maps all points of a figure such that each image point is on the opposite side of the reflecting line and the same distance from it as its preimage. If a point of the figure is on the reflecting line, then its image is the point itself.

Orientation

Every convex polygon has two orientations: clockwise and counterclockwise. The orientation assigned to a polygon depends on the direction of the path traced when moving along consecutive vertices. In Figure 8.6, $\triangle ABC$ has clockwise orientation while $\triangle A'B'C'$, its reflected image in line p, has counterclockwise orientation. A reflection, therefore, *reverses* orientation. It is this property of a reflection that makes the reflected image appear "backwards."

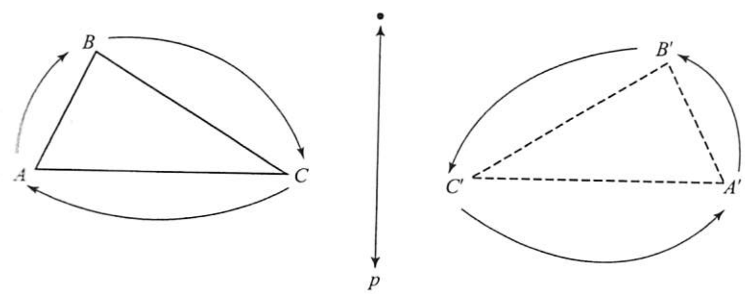


Figure 8.6 $\triangle ABC$ has clockwise orientation, while $\triangle A'B'C'$ has counterclockwise orientation.

Translations

A **translation** is an isometry that "slides" each point of a figure the same distance in the same direction, as illustrated in Figure 8.7. A translation preserves orientation.

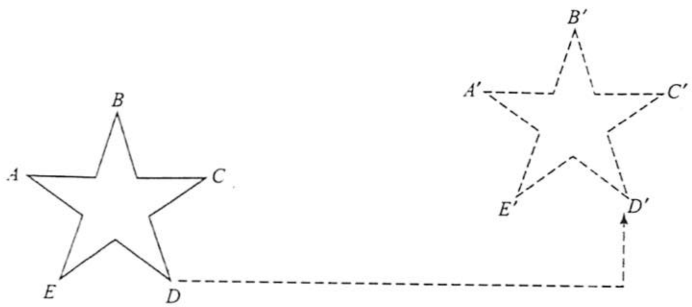


Figure 8.7 A translation or "slide" of an object.

Rotations

Suppose two identical pieces of paper have the same smiley face drawn in the same location. A pin is pushed through the papers when their edges are aligned. A rotation of the smiley face can be modeled by holding one paper fixed and turning the other paper, as illustrated in Figure 8.8. The pin represents the *center of rotation*.

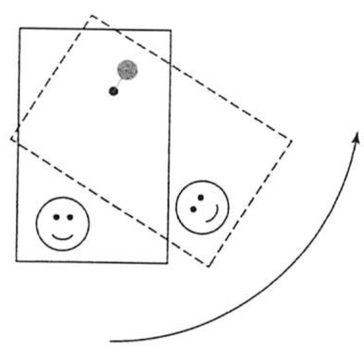


Figure 8.8 Modeling a rotation.

A **rotation** is an isometry that "turns" a figure through an angle about some fixed point called the **center of rotation**. Unless otherwise indicated, rotations are performed counterclockwise. Figure 8.9 shows a counterclockwise rotation of $\triangle ABC$ $x°$ about point O. The images of points A, B, and C are determined so that corresponding sides of the figure and its image have the same lengths and

$$m\angle AOA' = m\angle BOB' = m\angle COC' = x$$

The shorthand notation $R_{x°}(A) = A'$ indicates that the rotated image of point A after a counterclockwise rotation of $x°$ is point A'. You should verify that the rotation preserves orientation.

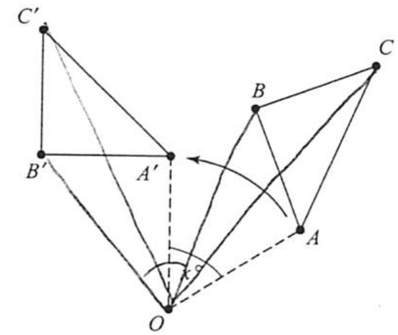


Figure 8.9 Counterclockwise rotation of $\triangle ABC$ $x°$ about point O.

Glide Reflection

There are only four types of isometries: reflections, translations, rotations, and *glide reflections*. A **glide reflection** is an isometry that combines a reflection over a line with a translation or "glide" in the direction parallel to the reflecting line, as in Figure 8.10. The line reflection and translation may be performed in either order. Reflections and glide reflections reverse orientation; translations and rotations have the same orientation.

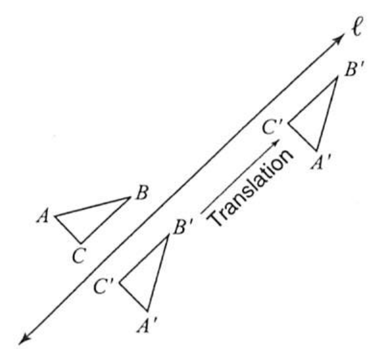


Figure 8.10 Glide reflection.

Classifying Isometries

Isometries are given special names according to whether they maintain or reverse orientation.

- A **direct isometry** is an isometry that preserves orientation. Translations and rotations are direct isometries.
- An **opposite isometry** is an isometry that reverses orientation. Line and glide reflections are opposite isometries.

MATH FACTS

If two geometric figures are related such that either one is the image of the other under some isometry, then the two figures are congruent.

Similarity Transformations

Not all transformations produce congruent images. When the image size setting of an office copying machine is set to a value other than 100 percent, the copy machine changes the size of the figure being reproduced without affecting its shape. This process is an example of a *dilation* in which the reproduced copy is the *dilated* image of the original. The image size setting represents the *scale factor* of the dilation. Figures 8.11 and 8.12 illustrate dilations with scale factors greater than 1 and less than 1.

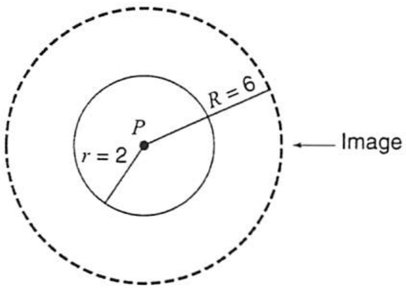


Figure 8.11 Dilation with center P of a circle with radius 2 using a scale factor of 3.

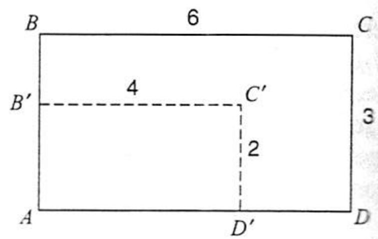


Figure 8.12 Dilation with center A of rectangle $ABCD$ using a scale factor of $\frac{2}{3}$.

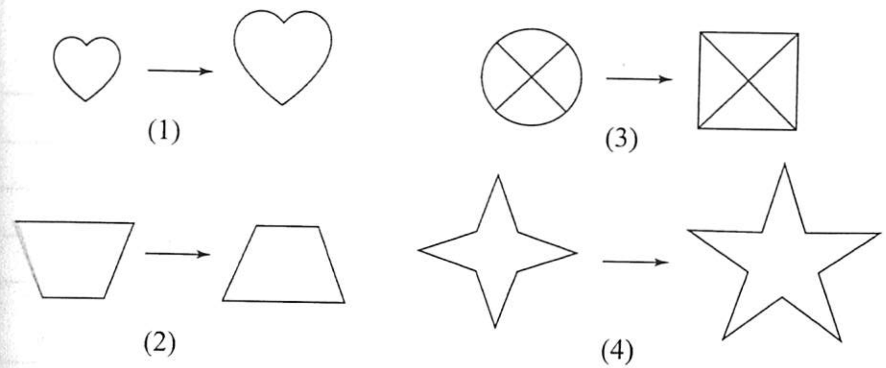
MATH FACTS

A **dilation** is a similarity transformation that changes the size of a figure by mapping each point onto its image such that the distance from the center of the dilation to the image is c times the distance from the center to the preimage. The multiplying factor c is called the **scale factor** or **constant of dilation**.

- If $c > 1$, the dilation enlarges the figure, as in Figure 8.11.
- If $0 < c < 1$, the dilatation shrinks the figure, as in Figure 8.12.

Example 1

Which transformation appears to represent an isometry?

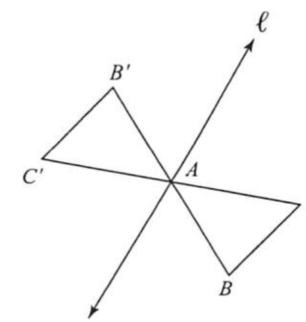


Solution: A transformation is an isometry if the preimage (original figure) and image are congruent. In choice (2), the pair of figures appear congruent, although the image is turned upside-down.

Example 2

Under what type of transformation, shown in the accompanying figure, is $\triangle AB'C'$ the image of $\triangle ABC$?

(1) dilation
(2) translation
(3) rotation about point A
(4) reflection in line ℓ



Solution: Consider each choice in turn:

- Choice (1): A dilation changes the size of the original figure. Since $\triangle AB'C'$ and $\triangle ABC$ are the same size, the figure does *not* represent a dilation.
- Choice (2): Since $\triangle AB'C'$ cannot be obtained by sliding $\triangle ABC$ in the horizontal (sideways) or vertical (up and down) direction, or in both directions, the figure does *not* represent a translation.
- Choice (3): A rotation about a fixed point "turns" a figure about that point. Since angles BAB' and CAC' are straight angles, $\triangle AB'C'$ is the image of $\triangle ABC$ after a rotation of 180° about point A.
- Choice (4): Since line ℓ is *not* the perpendicular bisector of $\overline{BB'}$ and $\overline{CC'}$, points B' and C' are not the reflected images of points B and C, respectively. Hence, the figure does *not* represent a reflection.

The correct choice is **(3)**.

Table 8.1 summarizes some key properties of transformations.

Table 8.1 Properties of Transformations

Type of Transformation	Preserves Distance	Preserves Angle Measure	Preserves Orientation	Type of Isometry
Reflection	✓	✓	✗	Opposite
Translation	✓	✓	✓	Direct
Rotation	✓	✓	✓	Direct
Glide reflection	✓	✓	✗	Qpposite
Dilation	✗	✓	✓	Not an Isometry

Check Your Understanding of Section 8.1

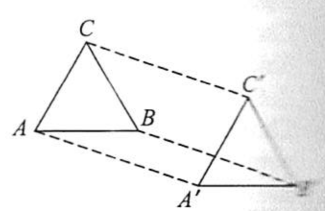
A. Multiple Choice

1. Which figure best represents a line reflection?

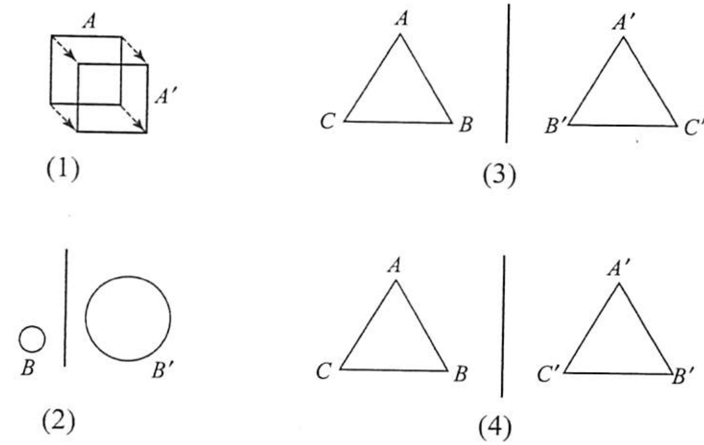
2. One function of a movie projector is to enlarge the image on the film. This procedure is an example of a
(1) dilation (2) reflection (3) rotation (4) translation

3. A reflection does *not* preserve
(1) collinearity
(2) segment measure
(3) orientation
(4) angle measure

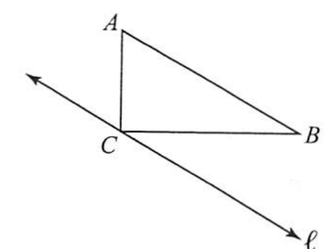
4. In the accompanying diagram, $\triangle A'B'C'$ is the image of $\triangle ABC$ under a transformation in which $\triangle ABC \cong \triangle A'B'C'$. This transformation is an example of a
(1) line reflection
(2) rotation
(3) translation
(4) dilation



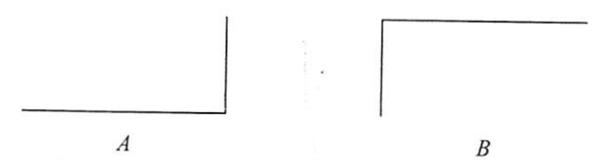
5. Ms. Brewer's art class is drawing reflected images. She wants her students to draw images reflected over a line. Which diagram represents a correctly drawn image?



6. In the accompanying diagram of right triangle ACB with the right angle at C, line ℓ is drawn through C and is parallel to $\overline{AB}$. If $\triangle ABC$ is reflected in line ℓ, forming the image $\triangle A'B'C'$, which statement is *not* true?
(1) C and C' are the same point.
(2) $m\angle ABC = m\angle A'B'C'$.
(3) Line ℓ is equidistant from A and A'.
(4) Point C is the midpoint of $\overline{AA'}$ area of $\triangle A'B'C'$.

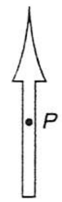


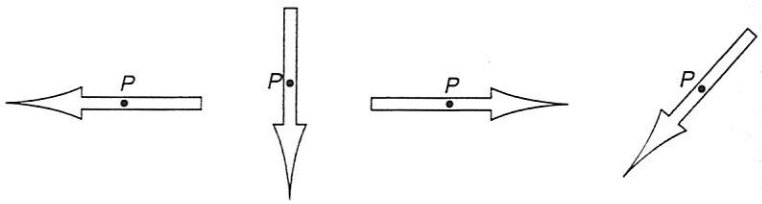
7. In the diagram, figure B is the image of figure A under which transformation?



(1) line reflection
(2) rotation
(3) translation
(4) dilation

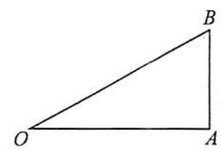
8. The accompanying diagram shows the starting position of the spinner on a board game. Which figure represents the image of this spinner after a 270° counterclockwise rotation about point P?

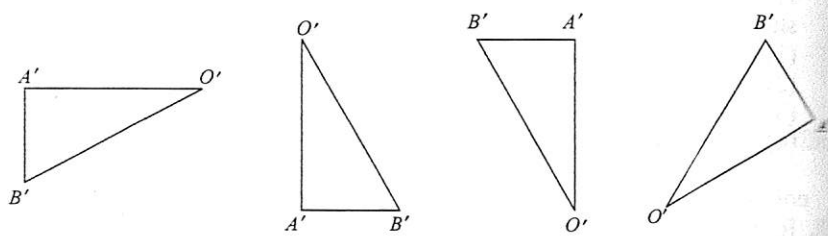




(1) (2) (3) (4)

9. If $\triangle OAB$, shown in the accompanying diagram, is rotated 90° *clockwise* about point O, which figure represents the image of this rotation?





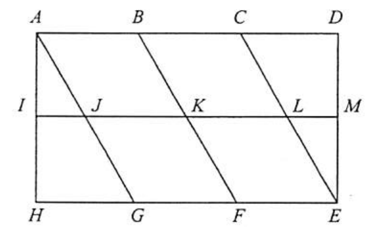
(1) (2) (3) (4)

10. The area of a circle is 16π square inches. After the circle is dilated, the circumference of the new circle is 16π inches. What is the scale factor?

(1) 1 (2) 2 (3) $\frac{1}{2}$ (4) $\frac{1}{4}$

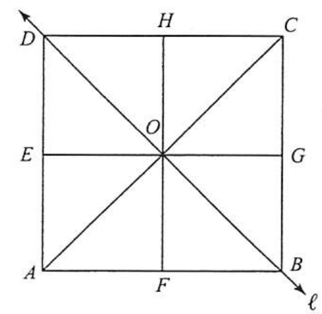
11. In the accompanying diagram, K is the image of A after a translation. Under the same translation, which point is the image of J?

(1) B
(2) C
(3) E
(4) F



B. *Show or explain how you arrived at your answer.*

12. Write the converse of "If a transformation is a line reflection, then it is an isometry." Give a counterexample to show that the converse is false.



13. In the accompanying diagram of square $ABCD$, F is the midpoint of $\overline{AB}$, G is the midpoint of $\overline{BC}$, H is the midpoint of $\overline{CD}$, and E is the midpoint of $\overline{DA}$.

a. Find the image of $\triangle EOA$ after it is reflected in line ℓ.
b. Is this isometry direct or opposite? Give a reason for your answer.

8.2 TYPES OF SYMMETRY

KEY IDEAS

There are many examples of *symmetry* in nature. People's faces, leaves, and butterflies have real or imaginary "lines of symmetry" that divide the figures into two parts that are mirror images. If a geometric shape has line symmetry, it can be "folded" along the line of symmetry so that the two parts coincide. A figure can also have *rotational* symmetry or *point* symmetry.

Line Symmetry

The objects in Figure 8.13 have *line symmetry*. A figure has line symmetry if it can be reflected in a line such that the image coincides with the preimage. The reflecting line is called the **line of symmetry** and divides the figure into two congruent reflected parts. The line of symmetry may be a horizontal line, a vertical line, or neither.

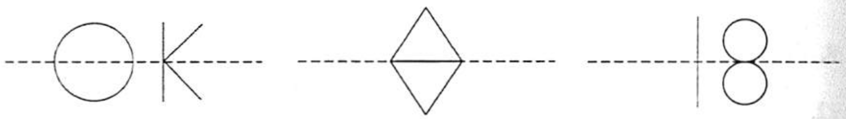


Figure 8.13 Horizontal line symmetry.

The objects in Figure 8.14 have vertical line symmetry.

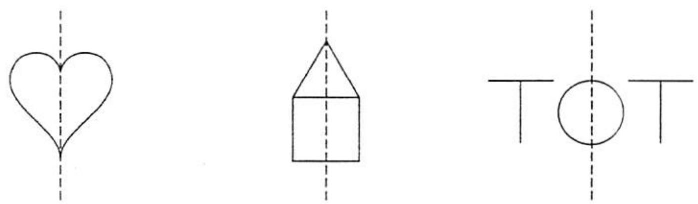


Figure 8.14 Vertical line symmetry.

Figure 8.15 illustrates that a figure may have both a horizontal and a vertical line of symmetry.

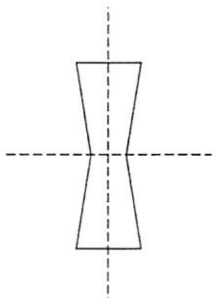


Figure 8.15 Both Horizontal and Vertical Line Symmetry.

As shown in Figure 8.16, a figure may have many lines of symmetry or may have no line of symmetry.

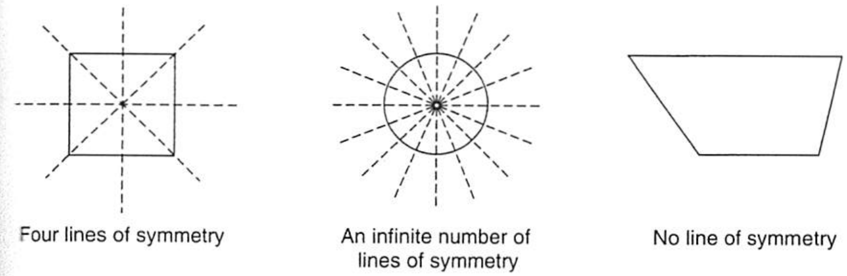


Figure 8.16 Figures with many or no lines of symmetry.

Rotational Symmetry

After a clockwise rotation of 60° about its center O, a regular hexagon will coincide with itself, as indicated in Figure 8.17. Regular hexagon $ABCDEF$ has 60° rotational symmetry.

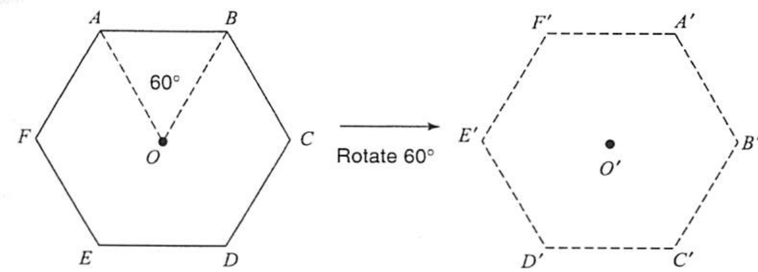


Figure 8.17 Rotational symmetry with a 60° angle of rotation.

A figure has **rotational symmetry** if it coincides with its image after a rotation through some positive angle less than 360°. Every regular polygon enjoys rotational symmetry about its center for an angle of rotation of $\frac{360°}{n}$, where n is the number of sides of the polygon. For an equilateral triangle, square, regular pentagon, and regular octagon, the angles of rotation are 120°, 90°, 72°, and 45°, respectively.

Point Symmetry

A figure has **point symmetry** with respect to a point when it has 180° rotational symmetry about that point, as in Figure 8.18.

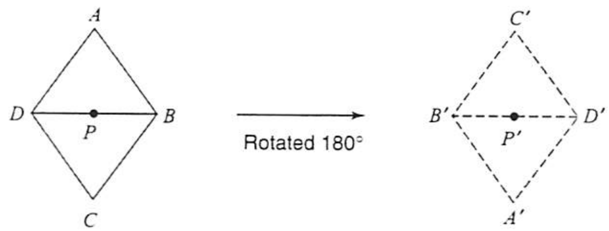


Figure 8.18 Point symmetry.

If you are not sure whether a figure has point symmetry, turn the page on which the figure is drawn upside down. Now compare the rotated figure with the original. If they look exactly the same, the figure has point symmetry.

Check Your Understanding of Section 8.2

A. Multiple Choice

1. If a rectangle is not a square, what is the greatest number of lines of symmetry that can be drawn?
   (1) 1 (2) 2 (3) 3 (4) 4

2. Which figure has one and only one line of symmetry?
   (1) rhombus (3) square
   (2) circle (4) isosceles triangle

3. Which type of symmetry, if any, does a square have?
   (1) line symmetry, only (3) both line and point symmetry
   (2) point symmetry, only (4) no symmetry

4. Which letter has both point and line symmetry?
   (1) **Z** (2) **T** (3) **C** (4) **H**

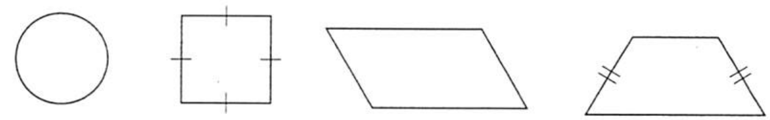
5. What is the total number of lines of symmetry for an equilateral triangle?
   (1) 1 (2) 2 (3) 3 (4) 4

6. Which letter has point symmetry but no line symmetry?
   (1) **E** (2) **S** (3) **W** (4) I

7. Which number has horizontal and vertical line symmetry?
   (1) **818** (2) **383** (3) **414** (4) **100**

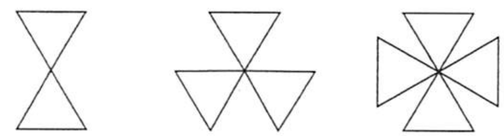
8. Which letter has line symmetry but no point symmetry?
   (1) **O** (2) **X** (3) **N** (4) **M**

9. Which geometric shape does *not* have any lines of symmetry?



(1) (2) (3) (4)

10. Which figures, if any, have *both* point symmetry and line symmetry?



A *B* *C*

(1) *A* and *C* only (3) none of the figures
(2) *B* and *C* only (4) all of the figures

8.3 TRANSFORMATIONS USING COORDINATES

KEY IDEAS

Transformations can be performed in the coordinate plane.

Reflections Using Coordinates

To reflect a *point* over a coordinate axis, flip it over the axis so that the image is the same distance from the reflecting line as the original point. In Figure 8.19:

- A' is the reflection of A over the x-axis. Point A' has the same x-coordinate as point A but the opposite y-coordinate.
- A'' is the reflection of A over the y-axis. Point A'' has the same y-coordinate as point A but the opposite x-coordinate.
- A''' is the reflection of A in the origin. The coordinates of point A''' are opposite those of point A.

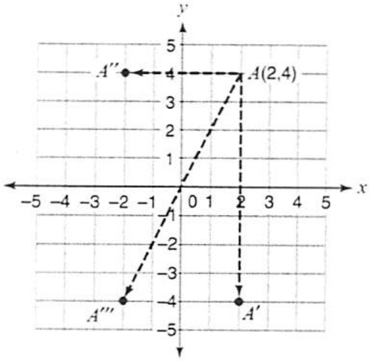


Figure 8.19 Reflecting over an axis and in the origin.

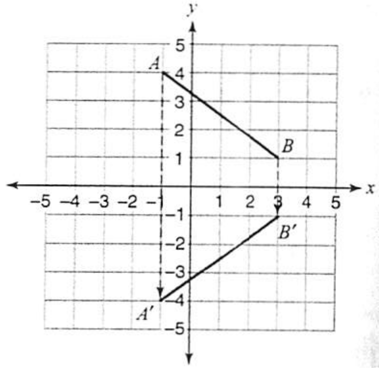


Figure 8.20 Reflecting a line segment.

The notation $r_{x\text{-axis}}(2, 4) = (2, -4)$ indicates that the reflected image of point (2, 4) over the x-axis is (2,–4). In general,

- $r_{x\text{-axis}}(x, y) = (x, -y)$
- $r_{y\text{-axis}}(x, y) = (-x, y)$
- $r_{\text{origin}}(x, y) = (-x, -y)$

To reflect a *line segment* over a coordinate axis, flip it over the axis by reflecting each endpoint of that segment. Then connect the images of the endpoints. If the endpoints of $\overline{AB}$ are $A(-1, 4)$ and $B(3, 1)$, then, after a reflection of $\overline{AB}$ over the x-axis, the image is $\overline{A'B'}$ with endpoints $A'(-1,-4)$ and $B'(3,-1)$. See Figure 8.20. To reflect a point over the line $y = x$ or over the line $y = -x$, as illustrated in Figure 8.21, use these rules:

- $r_{y=x}\, A(x, y) = A'(y, x)$
- $r_{y=-x}\, A(x, y) = A''(-y, -x)$

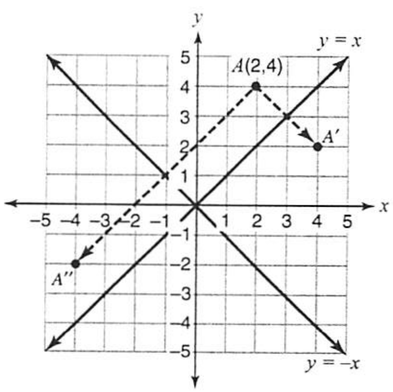
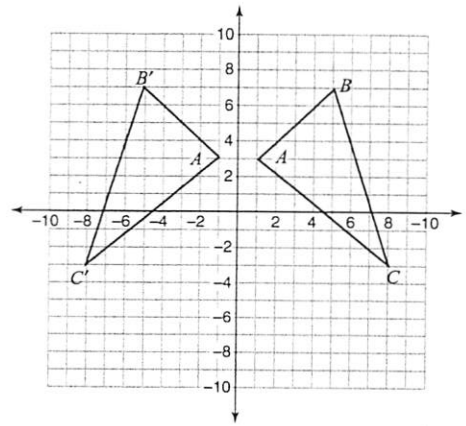


Figure 8.21 Reflecting over $y = \pm x$.

Example 1

Graph $\triangle ABC$ with coordinates $A(1, 3)$, $B(5, 7)$, and $C(8, -3)$. On the same set of axes graph $\triangle A'B'C'$, the reflection of $\triangle ABC$ over the y-axis.

Solution: After graphing $\triangle ABC$, reflect points A, B, and C over the y-axis, as shown in the accompanying figure. Then connect the image points A', B', and C' with line segments.



Reflection in a Point

A point may be reflected in the coordinate plane in a point other than the origin. Under a reflection in point P, the image of point A is point A' provided P is the midpoint of $\overline{AA'}$, as illustrated in Figure 8.22.

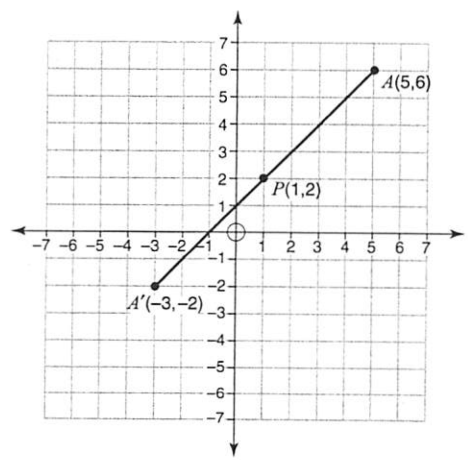


Figure 8.22 Points A and A' are reflections of each other in point P.

Translations Using Coordinates

In Figure 8.23, $\triangle A'B'C'$ is the image of $\triangle ABC$ under a translation that shifts each point of $\triangle ABC$ 7 units to the left and 2 units up. If (x, y) is any point of $\triangle ABC$, then $(x - 7, y + 2)$ represents the coordinates of the translated image point.

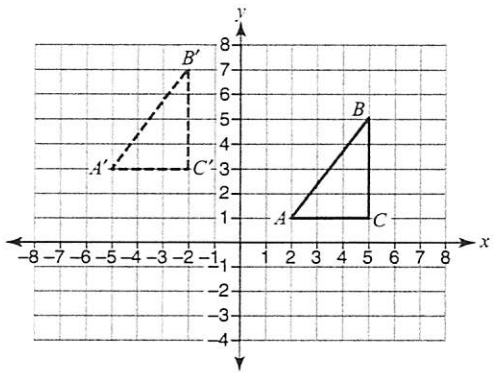


Figure 8.23 Translation of $\triangle ABC$.

The shorthand notation $T_{h,k}$ is sometimes used to represent a translation of a figure h units horizontally and k units vertically. The signs of h and k indicate direction: $h > 0$ means right, $h < 0$ is left; $k > 0$ represents up, and $k < 0$ is down. Referring again to Figure 8.23,

$$T_{-7,2}\, A(2, 1) = A'(2 + (-7), 1 + 2) = A'(-5, 3)$$

In general,

$$T_{h,k}\, P(x, y) = P'(x + h, y + k)$$

Example 2

The coordinates of the vertices of $\triangle ABC$ are $A(2, -3)$, $B(0, 4)$, and $C(-1, 5)$. If the image of point A under a translation is point $A'(0, 0)$, find the images of points B and C under the same translation.

Solution: In general, after a translation of h units in the horizontal direction and k units in the vertical direction, the image of $P(x, y)$ is $P'(x + h, y + k)$. Since

$$A(2, -3) \rightarrow A'(2 + h, -3 + k) = A'(0, 0)$$

it follows that

$$2 + h = 0 \quad \text{and} \quad h = -2$$
$$-3 + k = 0 \quad \text{and} \quad k = 3$$

Therefore:

$$B(0, 4) \rightarrow B'(0 + [-2], 4 + 3) = B'(2, 7)$$
$$C(-1, 5) \rightarrow C'(-1 + [-2], 5 + 3) = C'(-3, 8)$$

Rotations Using Coordinates

The notation $R_{x^\circ}(x, y)$ represents the counterclockwise rotation of point (x, y) through an angle of x°. Unless otherwise indicated, the center of rotation is the origin. In Figure 8.24, rectangle $AB'C'D'$ is the image of rectangle $ABCD$ under a 90° counterclockwise rotation about the origin. The vertices of rectangle $ABCD$ are mapped as follows:

$$R_{90^\circ} A(0, 0) = A(0, 0)$$
$$R_{90^\circ} B(6, 0) = B'(0, 6)$$
$$R_{90^\circ} C(6, 3) = C'(-3, 6)$$
$$R_{90^\circ} D(0, 3) = D'(-3, 0)$$

You should also verify that

$$R_{180^\circ} C(6, 3) = C'(-6, -3)$$
$$\text{and } R_{270^\circ} C(6, 3) = C'(-6, -3)$$

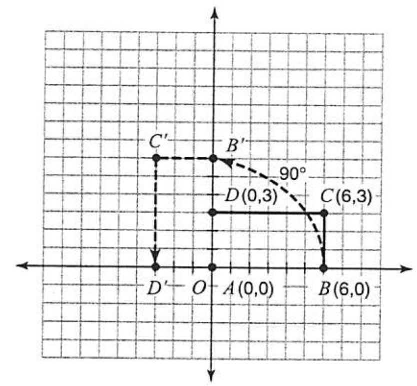


Figure 8.24 Rotation of a rectangle 90°.

MATH FACTS

After a rotation of 360°, the image of a figure coincides with itself so $R_{360^\circ}(x,y) = (x,y)$. To find the image of (x, y) after a counterclockwise rotation of 90°, or a multiple of 90°, use these rules:

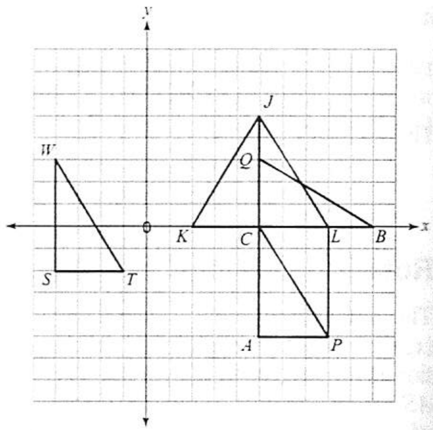
- $R_{90^\circ}(x, y) = (-y, x)$
- $R_{180^\circ}(x, y) = (-x, -y)$
- $R_{270^\circ}(x, y) = (y, -x)$

The notation $R_{-x^\circ}(P)$ represents a *clockwise* rotation of point P through an angle of x°. Thus, $R_{270^\circ}(P)$ and $R_{-90^\circ}(P)$ are equivalent transformations.

Example 3

In the accompanying figure, each grid box is 1 unit. Describe a transformation that maps:

a. $\triangle JCK$ onto $\triangle BCQ$
b. $\triangle WST$ onto $\triangle JCL$
c. $\triangle WST$ onto $\triangle JCK$
d. $\triangle JCK$ onto $\triangle CAP$



Solution:

a. A **rotation** of $\triangle JCK$ 270° counterclockwise about point C produces an image that coincides with $\triangle BCQ$. Thus, using C as the center of rotation,

$$R_{270^\circ}(\triangle JCK) = \triangle BCQ$$

b. A **translation** of 9 units horizontally and 2 units vertically will shift the vertices of $\triangle WST$ such that $W \to J$, $S \to C$, and $T \to L$. Thus,

$$T_{9,2}(\triangle WST) = \triangle JCL$$

c. A **glide reflection** comprised of a reflection of $\triangle WST$ in the y-axis and followed by a horizontal translation of the reflected triangle 1 unit to the right and 2 units up:

$$R_{y\text{-axis}}(\triangle WST = \triangle W'S'T') \text{ followed by } T_{1,2}(\triangle W'S'T') = \triangle JCK.$$

d. A **glide reflection** comprised of a reflection of $\triangle JCK$ in vertical segment followed by a vertical translation of its image, $\triangle JCL$, 5 units down:

$$r_{\overline{JCA}}(\triangle JCK) = \triangle JCL \text{ followed by } T_{0,-5}(\triangle JCL) = \triangle CAP$$

Dilations Using Coordinates

A dilation with a nonzero scale factor of k maps $P(x, y)$ onto $P'(kx, ky)$ where the origin is the center of the dilation. This transformation can be expressed using the notation

$$D_k(x, y) = (kx, ky)$$

Figure 8.25 illustrates a dilation in which $k > 1$ so that $OP' > OP$. To illustrate further, assume the coordinates of the endpoints of $\overline{AB}$ are $A(2, 4)$ and $B\left(\frac{1}{3}, 1\right)$.

The coordinates of the endpoints of $\overline{A'B'}$, the image of $\overline{AB}$ under a dilation with a scale factor of 3, are $A'(6, 12)$ and $B'(1, 3)$.

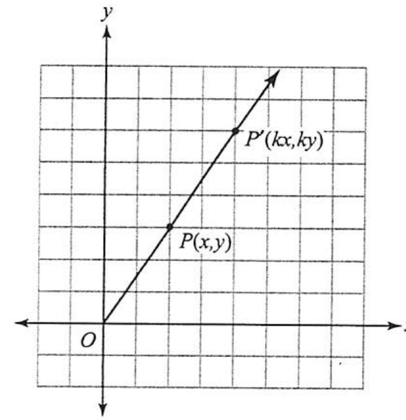


Figure 8.25 Dilation of point P.

Example 4

After a dilation with respect to the origin, the image of $A(2, 3)$ is $A'(4, 6)$. What are the coordinates of the point that is the image of $B(1, 5)$ after the same dilation?

Solution: Determine the constant of dilation. The constant of dilation is 2 since

$$A(2, 3) \to A'(\underline{2} \times 2, \underline{2} \times 3) = A'(4, 6)$$

Under the same dilation, the x– and y–coordinates of point B are also multiplied by 2:

$$D_2(1, 5) \to B'(\underline{2} \times 1, \underline{2} \times 5) = B'(2, 10)$$

Check Your Understanding of Section 8.3

A. *Multiple Choice*

1. Which transformation represents a dilation?
   (1) $(8, 4) \to (11, 7)$
   (2) $(8, 4) \to (-8, 4)$
   (3) $(8, 4) \to (-4, -8)$
   (4) $(8, 4) \to (4, 2)$

2. Which transformation is an example of an opposite isometry?
(1) $(x, y) \rightarrow (x + 3, y - 6)$
(2) $(x, y) \rightarrow (3x, 3y)$
(3) $(x, y) \rightarrow (y, x)$
(4) $(x, y) \rightarrow (y, -x)$

3. The image of point A after a dilation with a scale factor of 3 is (6, 15). What was the original location of point A?
(1) (2, 5) (2) (3, 12) (3) (9, 18) (4) (18, 45)

4. What is the image of point (–3, 4) under the translation that shifts (x, y) to $(x - 3, y + 2)$?
(1) (0, 6) (2) (6, 6) (3) (–6, 8) (4) (–6, 6)

5. The image of point (–2, 3) after a certain translation is (3, –1). What is the image of point (4, 2) after the same translation?
(1) (–1, 6) (2) (0, 7) (3) (5, 4) (4) (9, –2)

6. What is image of point (–3, –1) after a rotation of 90° about the origin?
(1) (3, 1) (2) (1, –3) (3) (3, –1) (4) (1, 3)

7. The three vertices of $\triangle ABC$ are located in Quadrant II. The image of $\triangle ABC$ after a reflection in the x-axis is $\triangle A'B'C'$. In which quadrant is the image of $\triangle A'B'C'$ located after a reflection in the y-axis?
(1) I (2) II (3) III (4) IV

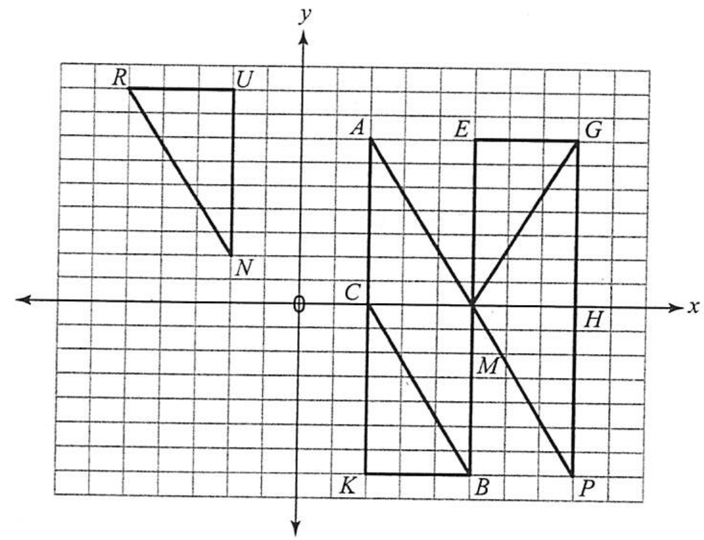
8. A function, f, is defined by the set {(2, 3), (4, 7), (–1, 5)}. If f is reflected in the line $y = x$, which point will be in the reflection?
(1) (–5, 1) (2) (5, –1) (3) (1, –5) (4) (–1, 5)

9. Which mapping rule does *not* represent an isometry in the coordinate plane?
(1) $(x, y) \rightarrow (2x, 2x)$
(2) $(x, y) \rightarrow (x + 2, y + 2)$
(3) $(x, y) \rightarrow (-x, y)$
(4) $(x, y) \rightarrow (x, -y)$

10. Point P' is the image of point $P(-3, 4)$ after a translation defined by $T_{(7, -1)}$. Which other transformation on P would also produce P' as its image?
(1) $r_{y=-x}$ (2) $r_{y\text{-axis}}$ (3) $R_{90°}$ (4) $R_{-90°}$

B. Show or explain how you arrived at your answer.

11–14. In the accompanying figure, each grid box is 1 unit. Identify each of the given transformations as either a reflection, translation, rotation, dilation, or glide reflection. State the reflection line, translation rule, center and angle of rotation, or the reflecting line and translation for a glide reflection.



11. $\triangle GHM \rightarrow \triangle ACM$

12. $\triangle BMC \rightarrow \triangle NUR$

13. $\triangle BKC \rightarrow \triangle GEM$

14. $\triangle ACM \rightarrow \triangle PHM$

15. If $T_{h,k}(2, -1) = (-2, 1)$, what are the coordinates of the image of (–3, 4) under the same translation?

16. If $T_{-2,3}(x, y) = (2, -1)$, what are the coordinates of the preimage point?

17. Under a reflection in point $X(1, 2)$, the image of point $P(3, -1)$ is P'. Determine the coordinates of point P'.

18. Carson is a decorator. He often sketches his room designs on the coordinate plane. He has graphed a square table on his grid so that its corners are at the coordinates $A(2, 6)$, $B(7, 8)$, $C(9, 3)$, and $D(4, 1)$. To graph a second identical table, he reflects $ABCD$ over the y-axis.

a. Using graph paper, sketch and label $ABCD$ and its image $A'B'C'D'$, which show the locations of the two tables.

b. Find the number of square units in the area of $ABCD$.

19. Given: Quadrilateral $ABCD$ with vertices $A(-2, 2)$, $B(8, -4)$, $C(6, -10)$, and $D(-4, -4)$. State the coordinates of $A'B'C'D'$, the image of quadrilateral $ABCD$ under a dilation of factor $\frac{1}{2}$. Prove that $A'B'C'D'$ is a parallelogram.

20. The vertices of $\triangle ABC$ are $A(-4, 7)$, $B(3, -2)$, and $C(8, -2)$. After a translation that maps (x, y) onto $(x + h, y + k)$, $\triangle A'B'C'$ is the image of $\triangle ABC$.

a. If $\triangle A'B'C'$ lies completely in Quadrant I, what are the smallest possible integer values of h and k?

b. How many square units are in the area of $\triangle ABC$?

c. If $\triangle A''B''C''$ is the image of $\triangle ABC$ after a dilation using a scale factor of 2 with respect to the origin, how many square units are in the area of $\triangle A''B''C''$?

21. The vertices of $\triangle PEN$ are $P(1, 2)$, $E(3, 0)$, and $N(6, 4)$. On graph paper draw and label $\triangle PEN$.

a. Graph and state the coordinates of $\triangle P'E'N'$, the image of $\triangle PEN$ after a reflection in the y-axis.

b. Graph and state the coordinates of $\triangle P''E''N''$, the image of $\triangle PEN$ under the translation $(x, y) \rightarrow (x + 4, y - 3)$.

22. Triangle SAM has coordinates $S(3, 4)$, $A(3, -5)$, and $M(-4, -2)$. On graph paper, graph and label $\triangle SAM$.

a. Graph and label $\triangle S'A'M'$, the image of $\triangle SAM$ after a reflection in the line $y = x$.

b. Graph and label $\triangle S''A''M''$, the image of $\triangle SAM$ after a dilation of 2. Express in simplest form the ratio of the area of $\triangle SAM$ to the area of $\triangle S''A''M''$.

8.4 COMPOSING TRANSFORMATIONS

KEY IDEAS

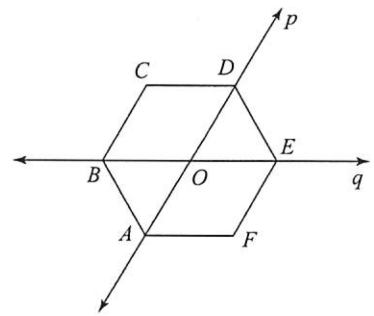
A glide reflection is an example of a *composite transformation* as it combines two other transformations to form a new transformation. When evaluating a composite transformation, the order in which the transformations are performed matters.

Composite Transformations

A **composite transformation** is a series of transformations, one followed by the other, in which the image of one transformation is used as the preimage of the next transformation.

Example 1

In the accompanying figure, p and q are lines of symmetry for regular hexagon $ABCDEF$ intersecting at point O, the center of the hexagon. Determine the final image of the composite transformation of the reflection of $\overline{AB}$ in line q followed by a reflection of its image in line p.



Solution: Since $r_{\text{line } q}(\overline{AB}) = \overline{CB}$ and $r_{\text{line } p}(\overline{CB}) = \overline{EF}$, the final image is $\overline{EF}$.

Example 2

The coordinates of the vertices of $\triangle ABC$ are $A(2, 0)$, $B(1, 7)$, and $C(5, 1)$.

a. Graph $\triangle A'B'C'$ the reflection of $\triangle ABC$ over the y-axis and graph $\triangle A''B''C''$ the reflection of $\triangle A'B'C'$ over the x-axis.

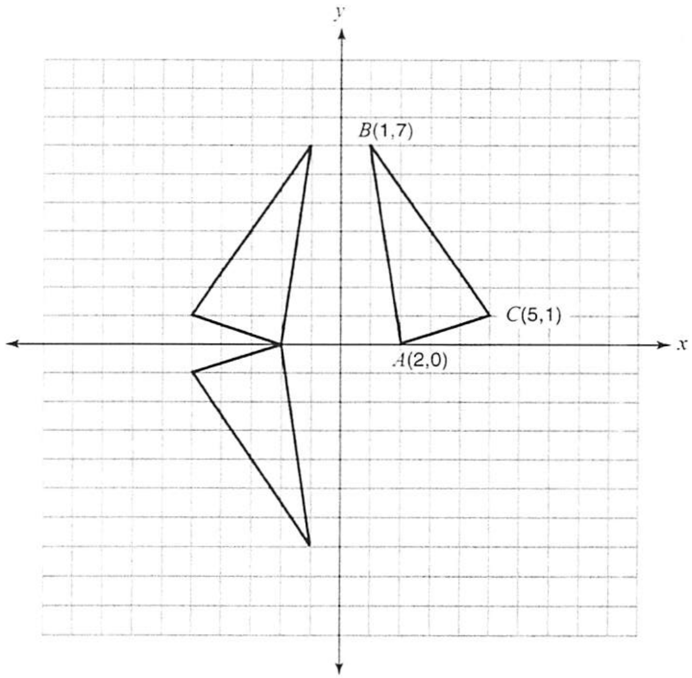
b. What single type of transformation maps $\triangle ABC$ onto $\triangle A''B''C''$?

Solution:

a. Under the given composite transformation

$$A\ (2, 0) \rightarrow A'\ (-2, 0) \rightarrow A''\ (-2, 0)$$
$$B\ (1, 7) \rightarrow B'\ (-1, 7) \rightarrow B''\ (-1, 7)$$
$$C\ (5, 1) \rightarrow C'\ (-5, 1) \rightarrow C''\ (-5, 1)$$

See the accompanying figure.



b. $R_{180°}(\triangle ABC) = \triangle A''B''C''$.

Composite Notation

A composite transformation is indicated by placing a centered circle between two transformations, as in $r_{x\text{-axis}} \circ T_{1,2}(-1, 4)$. The transformation on the right side of the centered circle is always evaluated first. Thus, $r_{x\text{-axis}} \circ T_{1,2}(-1, 4)$ represents a composite transformation consisting of a translation followed by a reflection. To find the final image point,

- First perform the translation:

$$T_{1,2}(-1, 4) = (-1 + 1, 4 + 2) = (0, 6)$$

- Then reflect the translated image point:

$$r_{x\text{-}axis}(0, 6) = (0, -6)$$

Thus, $r_{x\text{-axis}} \circ T_{1,2}(-1, 4) = (0, -6)$. The composite transformation $r_{x\text{-axis}} \circ T_{1,2}(-1, 4)$ can also be written as $r_{x\text{-axis}}(T_{1,2}(-1, 4))$ where the transformation enclosed by the parentheses is performed first.

Example 3

Using the same figure as in Example 1, identify the image of the composite transformation $r_q \circ R_{60°}(E)$.

Solution: The notation $r_q \circ R_{60°}(E)$ represents the composite transformation of a 60° counterclockwise rotation of point E followed by a reflection of the rotated image point in line q. Because, $R_{60°}(E) = D$,

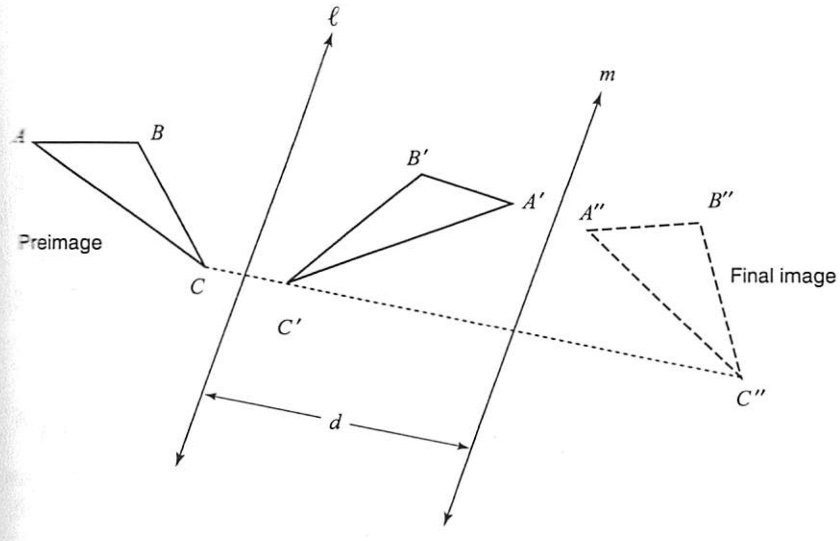
$$r_q \circ R_{60°}(E) = r_q(D) = F$$

The final image point is $\boldsymbol{F}$.

Composing Reflections over Two Lines

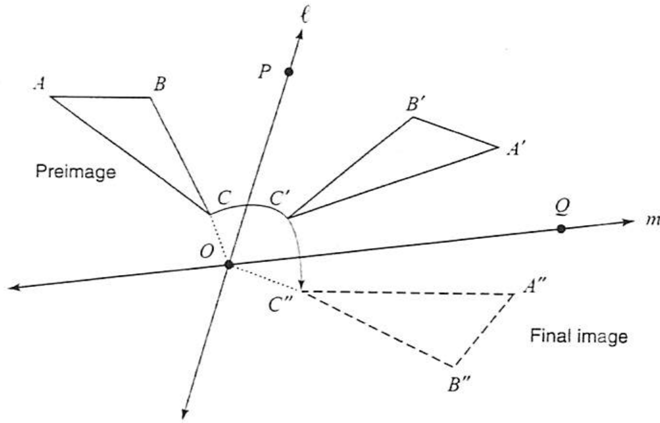
The composition of two rotations with the same center is a *rotation*. The composition of two translations is a *translation*. The composition of two reflections, however, is *not* a reflection. There are two possibilities to consider:

- Composing two reflections over parallel lines *translates* the original figure, as shown in Figure 8.26.
- Composing two reflections over intersecting lines *rotates* the original figure, as shown in Figure 8.27.



Length of $\overline{CC''} = 2 \times d$

Figure 8.26 Composing reflections over two parallel lines.



$m\angle COC'' = 2 \times m\angle POQ$

Figure 8.27 Composing reflections over two lines intersecting at point *O*.

MATH FACTS

Reflection–Reflection Theorem

Case 1: Reflecting over Parallel Lines
The composition of two reflections over two parallel lines is a translation.

- The direction of the translation is perpendicular to the reflecting lines.
- The distance between the final image and the preimage is two times the distance between the parallel lines.

Case 2: Reflecting over Intersecting Lines
The composition of two reflections over two intersecting lines is a rotation about their point of intersection. The angle of rotation is equal to two times the measure of the angle formed by the reflecting lines at their point of intersection.

Composite Transformations and Congruent Figures

Here are some other observations about composite transformations that you should know:

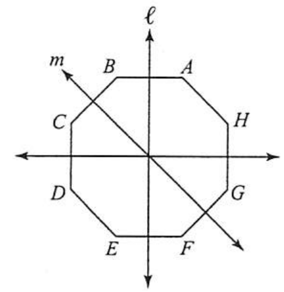
- Because of the Reflection–Reflection Theorem, any translation or rotation can be expressed as the composition of two reflections.
- If two figures are congruent, there exists a transformation that maps one figure onto the other.
- In a plane, one of two congruent figures can be mapped onto the other by a composition of at most three reflections.

Check Your Understanding of Section 8.4

A. Multiple Choice

1. The composition of two equal glide reflections is equivalent to
(1) a translation that is twice the distance of a single glide reflection
(2) a dilation with a scale factor of 2
(3) a rotation
(4) a reflection in a line perpendicular to the direction of the translation.

2–5. In the accompanying diagram of regular octagon *ABCDEFGH*, lines ℓ and *m* are lines of symmetry.



Exercises 2–5

2. What is the image of the reflection of $\overline{AB}$ over line *m* followed by a reflection of the image over line ℓ?
(1) $\overline{CD}$ (2) $\overline{AH}$ (3) $\overline{HG}$ (4) $\overline{FG}$

3. What is the image of a 135° counterclockwise rotation of point *H* followed by a reflection of the rotated image over line *m*?
(1) *B* (2) *D* (3) *E* (4) *G*

4. What is $r_m \circ r_\ell\,(\overline{FG})$?
(1) $\overline{CD}$ (2) $\overline{AH}$ (3) $\overline{HG}$ (4) $\overline{BC}$

5. What is $r_m \circ r_\ell \circ r_m\,(H)$?
(1) *A* (2) *E* (3) *F* (4) *G*

6. What are the coordinates of $r_{y=x} \circ r_{x\text{-axis}}\,(3, 1)$?
(1) (1, −3) (2) (−3, 1) (3) (−1, 3) (4) (3, −1)

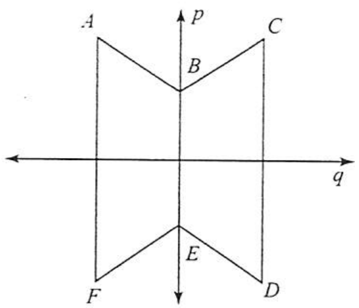
7. What are the coordinates of $r_{x\text{-axis}} \circ R_{90°}$ (4, –2)?
(1) (2, 4) (2) (2,–4) (3) (4, 2) (4) (–4, 2)

8. The coordinates of $\triangle JRB$ are $J(1, -2)$, $R(-3, 6)$, and $B(4, 5)$. What are the coordinates of the vertices of its image after the transformation $T_{2,-1} \circ r_{y\text{-axis}}$?
(1) (3, 1), (–1, –7), (6, –6)
(2) (3, –3), (–1, 5), (6, 4)
(3) (1, –3), (5, 5), (–2, 4)
(4) (–1, –2), (3, 6), (–4, 5)

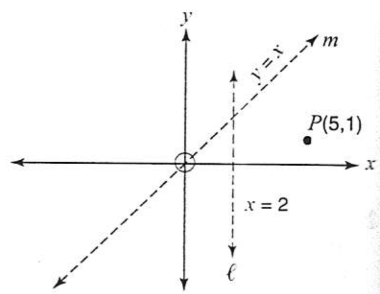
9–10. In the accompanying diagram, p and q are lines of symmetry for figure $ABCDEF$.

9. What is $r_p \circ r_q \circ r_p (A)$?
(1) B (2) D (3) E (4) F

10. What is $r_q \circ r_p \circ r_q (\overline{BC})$?
(1) $\overline{AB}$ (2) $\overline{BC}$ (3) $\overline{DE}$ (4) $\overline{EF}$



Exercises 9–10



Exercises 11–12

11–12. In the accompanying diagram, the equation of line ℓ is $x = 2$ and the equation of line m is $y = x$.

11. What are the coordinates of the image of $r_\ell \circ r_m (P)$?
(1) (3, 5) (2) (–1, 1) (3) (3, –5) (4) (1, –1)

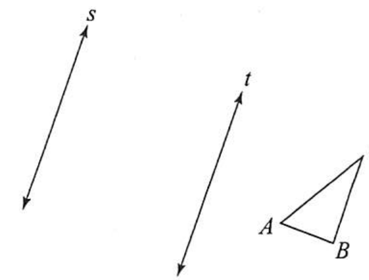
12. What are the coordinates of the image of $r_{\text{origin}} \circ r_\ell \circ r_{x\text{-axis}} (P)$?
(1) (–1, 1) (2) (1, 1) (3) (3, 5) (4) (–5, –3)

13. What are the coordinates of the image of point $A(-4, 1)$ after the composite transformation $R_{90°} \circ r_{y=x}$ where the origin is the center of rotation?
(1) (–1, –4) (2) (–4, –1) (3) (1, 4) (4) (4, 1)

14. If the coordinates of point A are (–2, 3), what is the image of A under the composite transformation $r_{y\text{-axis}} \circ D_3$?
(1) (–6, –9) (2) (9, –6) (3) (5, 6) (4) (6, 9)

15. The composite transformation that reflects point P through the origin, the x-axis, and the line $y = x$, in the order given, is equivalent to which rotation of point P about the origin?
(1) $R_{90°}$ (2) $R_{180°}$ (3) $R_{270°}$ (4) $R_{360°}$

16. In the accompanying diagram, $s \parallel t$. Which is equivalent to the composition of line reflections $r_s \circ r_t (\triangle ABC)$?
(1) a rotation
(2) a line reflection
(3) a translation
(4) a glide reflection



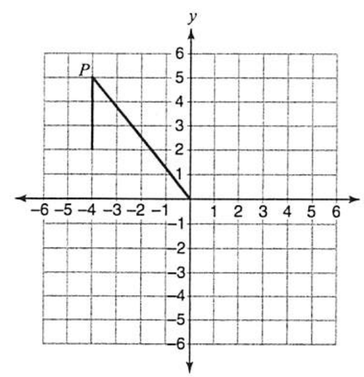
17. Given the transformations: $R(x, y) \to (-x, y)$ and $S(x, y) \to (y, x)$. What is $(R \circ S)(5, -1)$?
(1) (1, 5) (2) (1, –5) (3) (–1, 5) (4) (–1, –5)

18. Which transformation is equivalent to the composite line reflections $r_{y\text{-axis}} \circ r_{y=x} (\overline{AB})$?
(1) a rotation (3) a translation
(2) a dilation (4) a glide reflection

B. *Show or explain how you arrived at your answer.*

19. Given point $A(-2, 3)$. State the coordinates of the image of A under the composite transformation $T_{-3,4} \circ r_{x\text{-axis}}$.

20. Draw the image of the figure shown in the accompanying grid after the composite transformation $r_{y\text{-axis}} \circ R_{90°}$. State the coordinates of P', the image of point P under this transformation.



Exercise 20

21. Using the same figure as for Exercises 11–14 in Section 8.3, state the composite transformation rule that maps $\triangle RUN$ onto $\triangle GEM$.

22. If (–4, 8) is the image under the composite transformation $T_{h,3} \circ T_{-2,k}(-3, 0)$, what are the coordinates of the image of (2, –1) under the same composite transformation?

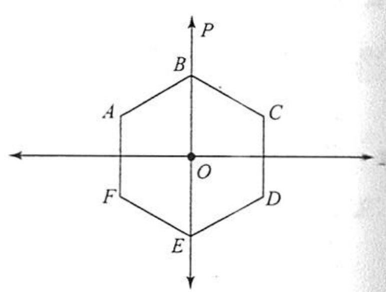
23. Given $A(0, 5)$ and $B(2, 0)$, graph and label $\overline{AB}$. Under the transformation $r_{x\text{-axis}} \circ r_{y\text{-axis}}(\overline{AB})$, A maps to A'', and B maps to B''. Graph and label $\overline{A''B''}$. What single transformation would map $\overline{AB}$ to $\overline{A''B''}$?

24. The coordinates of the vertices of $\triangle ABC$ are $A(-1, 2)$, $B(6, 2)$, and $C(3, 4)$. On graph paper, draw and label $\triangle ABC$.
a. On graph paper, graph and state the coordinates of $\triangle A'B'C'$, the image of $\triangle ABC$ after the composition $R_{90°} \circ r_{x\text{-axis}}$.
b. Write a transformation equivalent to $R_{90°} \circ r_{x\text{-axis}}$.

25. a. On graph paper, graph and label the triangle whose vertices are $A(0, 0)$, $B(8, 1)$, and $C(8, 4)$. Then graph, label, and state the coordinates of $\triangle A'B'C'$, the image of $\triangle ABC$ under the composite transformation $r_{x=0} \circ r_{y=x}$ ($\triangle ABC$). Which single transformation maps $\triangle ABC$ onto $\triangle A'B'C'$
(1) rotation (3) glide reflection
(2) dilation (4) translation

b. On the same set of axes, graph, label, and state the coordinates of $\triangle A''B''C''$, the image of $\triangle ABC$ under the composite transformation $r_{y=2} \circ r_{y=0}$ ($\triangle ABC$). Which single transformation maps $\triangle ABC$ onto $\triangle A''B''C''$?
(1) rotation (3) glide reflection
(2) dilation (4) translation

26. In the accompanying diagram of regular hexagon $ABCDEF$ with center O, L and P are lines of symmetry. State the final image under each composite transformation.
a. $r_P \circ R_{-120°}(C)$
b. $r_L \circ r_P(\overline{AB})$
c. $R_{60°} \circ r_O(A)$



CHAPTER 9

LOCUS AND COORDINATES

9.1 SIMPLE LOCUS

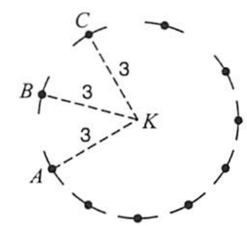
KEY IDEAS

A **locus** may be thought of as a path consisting of the set of all points, and only those points, that satisfy a particular condition. A locus that satisfies only one condition, called a **simple locus**, takes the form of a line, a pair of lines, or a curve such as a circle. The plural of locus is *loci*.

Finding a Simple Locus: An Example

To find the locus of points that are 3 units from point K:

- Make a diagram. Keep drawing points 3 units from point K until you discover a pattern.
- Connect the points with a broken curve as shown in the accompanying figure.
- Write a sentence that describes what you have discovered: "The locus of points that are 3 units from point K is a circle that has point K as its center and a radius of 3 units."



Five Basic Loci

The accompanying table summarizes the five basic loci that you need to know.