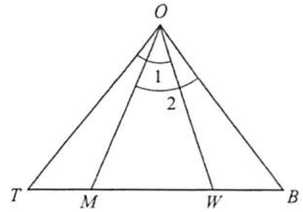
4. Given the statement: "A triangle cannot have two right angles." In order to prove this statement by the indirect method, it should be assumed that a triangle
(1) does not have a right angle
(2) has two right angles
(3) has one right angle
(4) does not have two right angles

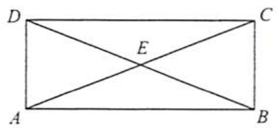
B. *Write a two-column proof.*

5. Given: $\angle 1 \cong \angle 2$.
Prove: $\angle TOM \cong \angle BOW$.



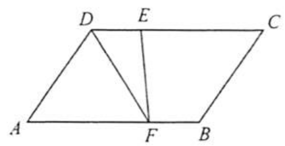
Exercise 5

6. Given: $\overline{AC}$ and $\overline{BD}$ bisect each other at E, $\overline{AC} \cong \overline{BD}$.
Prove: $\triangle AED$ is isosceles.



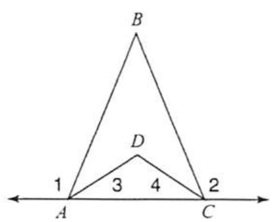
Exercise 6

7. Given: $\overline{AB} \cong \overline{CD}$, $\overline{DE} \cong \overline{BF}$, $\triangle ADF$ is equilateral.
Prove: $\overline{CE} \cong \overline{AD}$.



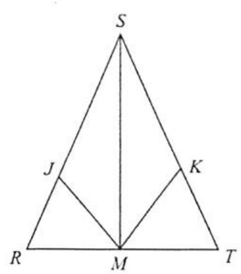
Exercise 7

8. Given: $\angle 1 \cong \angle 2$, $\overline{AD}$ bisects $\angle BAC$, $\overline{CD}$ bisects $\angle BCA$.
Prove: $\angle 3 \cong \angle 4$.



Exercise 8

9. Given: $\overline{SJ} \cong \overline{SK}$, $\overline{JR} \cong \overline{MR}$, $\overline{KT} \cong \overline{MT}$, M is the midpoint of $\overline{RT}$.
Prove: $\overline{SR} \cong \overline{ST}$.



Exercise 9

1.5 LOGICAL STATEMENTS

KEY IDEAS

A **statement** is a sentence that can be judged as either true or false, but not both. The truth (T) or falsity (F) of a statement is its **truth value**. Statements such as "I like milk" and "I do *not* like milk" are negations of each other and have opposite truth values. Two statements can be joined by *AND* or *OR* to form a **compound** statement.

Statements and Their Negations

To form the **negation** of a statement, insert the word NOT so that the original statement and its negation have opposite truth values.

STATEMENT: The capital of New York State is Buffalo. [FALSE]

NEGATION: The capital of New York State is not Buffalo. [TRUE]

or

It is not true that the capital of New York State is Buffalo.

Example 1

Write the negation of each statement.

a. Parallel lines do not intersect.
b. $m\angle A > 35$

Solution: a. Parallel lines intersect.
b. $m\angle A \ngtr 35$ or $m\angle A \leq 35$

Symbolic Notation for Negation

Sometimes it is convenient to use a shorthand notation in which letters such as p and q serve as placeholders for actual statements. The negation of p is denoted by $\sim p$ and is read as "not p." If p is the statement, "Monday is the day after Sunday," then $\sim p$ represents, "Monday is *not* the day after Sunday." Because a statement and its negation always have opposite truth values, it is not possible for p and $\sim p$ to have the same truth value.

Connecting Statements with *AND* or *OR*

Two different statements may be combined using the word AND or OR to form a new statement.

- A **conjunction** uses *AND* to connect two statements. The conjunction "It is spring *and* the birds are chirping" is true only when the two statements joined by the word AND are both true. Each of the statements that make up a conjunction is called a **conjunct**. The symbol for conjunction is $\wedge$. The symbolic notation $p \wedge q$ represents the conjunction of statements p and q.
- A **disjunction** uses *OR* to connect two statements. The disjunction "I will study *or* I will go to the movies" is true only when at least one of the statements joined by the word *or* is true. Each of the statements that make up a disjunction is called a **disjunct**. The symbol for disjunction is $\vee$. The symbolic notation $p \vee q$ represents the disjunction of statements p or q.

Truth Tables

A **truth table** summarizes the truth values a compound statement takes on for all possible combinations of truth values of the simple statements it comprises. In each of the accompanying truth tables, the first two columns give the possible combinations of truth values for statements p and q. On each row of the truth table, the last column shows the truth value for the compound statement using the truth values of p and q on that same row.

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

Example 2

Let p represent "It is cold in January" and q represent "It snows in August." Express each of the following statements in symbolic form.

a. It is cold in January or it does not snow in August.
b. It is not cold in January and it does not snow in August.
c. It is not true that it is cold in January and it snows in August.

Solution: a. Write the disjunction of p and the negation of q:

$$p \vee \sim q$$

b. Write the conjunction of the negation of p and the negation of q:

$$\sim p \wedge \sim q$$

c. Write the negation of the conjunction of p and q:

$$\sim (p \wedge q)$$

Example 3

Let p represent "All right angles are congruent" and q represent "Vertical angles are always supplementary." Determine the truth value of each compound statement:

a. $\sim p \vee q$ b. $p \wedge \sim q$ c. $\sim p \vee \sim q$ d. $\sim (p \wedge q)$

Solution: Statement p is true and statement q is false since vertical angles are always congruent. Substitute these truth values into each logic statement:

a. $p \vee \sim q$
$\sim(T) \vee F$
$F \vee F$
False

b. $p \wedge \sim q$
$T \wedge \sim(F)$
$T \wedge T$
True

c. $\sim p \vee \sim q$
$\sim(T) \vee \sim(F)$
$F \vee T$
True

d. $\sim(p \wedge q)$
$\sim(T \wedge F)$
$\sim(F)$
True

Drawing Inferences

Based on the truth value of a compound statement, you may be able to draw a conclusion about the truth value of a related statement.

Example 4

Given: "Ben has a driver's license or Ben is not 18 years old " is false. What is the truth value of "Ben is 18 years old"?

Solution:

- Because the given disjunction is false, both disjuncts must be false.
- Hence, the truth value of the statement "Ben is not 18 years old" is false.
- Since a statement and its negation have opposite truth values, the truth value of the statement "Ben is 18 years old" is **true.**

Example 5

Given: "I will buy a new suit or I will not go to the dance" is true and "I will buy a new suit" is false. What is the truth value of the statement, "I will go to the dance"?

Solution:

- Because the given disjunction is true, at least one of the two disjuncts must also be true.
- It is given that the disjunct "I will buy a new suit" is false. This means that the disjunct "I will not go to the dance" is true.
- Since a statement and its negation have opposite truth values, the truth value of "I will go to the dance" is **false**.

Check Your Understanding of Section 1.5

A. Multiple Choice.

1. Let p represent "It is cold" and let q represent "It is snowing." Which expression can be used to represent "It is cold and it is not snowing"?
(1) $\sim p \wedge q$ (2) $p \wedge \sim q$ (3) $p \vee \sim q$ (4) $\sim p \vee q$

2. Let p represent "The figure is a triangle" and q represent "The figure does not contain two obtuse angles." Which expression can be used to represent "The figure is not a triangle and the figure contains two obtuse angles."
(1) $\sim p \wedge q$ (2) $p \wedge \sim q$ (3) $\sim p \wedge \sim q$ (4) $\sim(p \wedge q)$

3. If p represents "Math is fun" and q represents "Math is difficult," which expression can be used to represent "It is not true that Math is not fun or math is difficult."
(1) $\sim(p \vee q)$ (2) $\sim(p \wedge q)$ (3) $\sim p \vee q$ (4) $\sim(\sim p \vee q)$

4. Which statement is always false?
(1) $p \vee \sim q$ (2) $\sim p \wedge q$ (3) $\sim p \vee p$ (4) $q \wedge \sim q$

5. Given the true statements: "Jason goes shopping or he goes to the movies" and "Jason does not go to the movies." Which statement must also be true?
(1) Jason stays home.
(2) Jason goes shopping.
(3) Jason does not go shopping.
(4) Jason does not go shopping and he does not go to the movies.

6. If $p \wedge \sim q$ is true, then which is true?
(1) p and q are both true. (3) p is true and q is false.
(2) p is false and q is true. (4) p and q are both false.

B. Show or explain how you arrived at your answer.

7. Let x represent "Mr. Ladd teaches mathematics" and let y represent "Mr. Ladd is the football coach." Write in symbolic form: "Mr. Ladd does not teach mathematics and Mr. Ladd is the football coach."

8. Let p represent the statement "Perpendicular lines intersect at right angles" and q represents the statement "Parallel lines do not intersect." Determine the truth value of each compound statement:
a. $p \wedge \sim q$ b. $\sim p \vee q$ c. $\sim p \wedge \sim q$
d. $\sim(p \vee q)$ e. $\sim(\sim p \wedge \sim q)$

9. Each part that follows consists of a set of three sentences. The truth values of the first two sentences are given. Determine the truth value of the third sentence.

a. It rains or it is cold. It is cold. It rains.	TRUE FALSE ?
b. The month is June and it is *not* warm. The month is June. It is warm.	FALSE TRUE ?
c. I will study or I will *not* pass the test. I will not study. I will pass the test.	TRUE ? TRUE
d. I will *not* work at camp this summer or I will attend summer school. I will work at camp this summer. I will not attend summer school.	FALSE TRUE ?
e. The month is *not* January and it is *not* snowing. The month is January. It is not snowing.	? TRUE TRUE

10. Let p represent "I go to the beach" and q represent "I get a sunburn."

a. Using p and q, write the following two statements in symbolic form:
   (1) It is not the case that I went to the beach and I got a sunburn.
   (2) I did not go to the beach or I did not get a sunburn.

b. In the last two columns of the accompanying truth table, enter the symbolic forms of statements (1) and (2) determined in part a. Then complete the truth table.

		Symbolic Form of Statement (1)	Symbolic Form of Statement (2)
p	q		
T	T		
T	F		
F	T		
F	F		

c. Determine if statements (1) and (2), written in part a, are logically equivalent. Justify your answer.

1.6 CONDITIONAL STATEMENTS

KEY IDEAS

An If–Then statement is called a **conditional statement**. Associated with each conditional statement are three other conditional statements: the *converse*, the *inverse*, and the *contrapositive*. Certain pairs of these conditionals always agree in their truth values and, as a result, are **logically equivalent**.

Truth Value of a Conditional

A **conditional statement** is a statement that has the form, "If p, then q," as in

If I live in Albany (hypothesis), then I am a New Yorker (conclusion)."

The "If" part of a conditional statement is the **hypothesis** and the part that follows "then" is the **conclusion**. A conditional statement is false when there is at least one situation for which the hypothesis is true, but the conclusion is

false. Such a situation is called a **counterexample**. Consider the conditional statement

"If the month has 31 days, then it is winter."

To prove that the conditional statement is false, it is only necessary to find a single counterexample. The month of July, which has 31 days, serves as a counterexample.

MATH FACTS

A conditional statement is true *except* in the single instance when the hypothesis is true and the conclusion is false. A specific situation for which this occurs is called a **counterexample**.

Forming Related Conditional Statements

By interchanging or negating both parts of a conditional statement, or by doing both, three related conditional statements can be formed.

Type of Statement	Forming a Conditional from "If p, then q"
Converse	Interchange p and q: "If q, then p"
Inverse	Negate both p and q: "If $\sim p$, then $\sim q$"
Contrapositive	Interchange and negate both p and q: "If $\sim q$, then $\sim p$"

Here is an example:

ORIGINAL:	If I live in Albany, then I am a New Yorker.	(TRUE)
CONVERSE:	If I am a New Yorker, then I live in Albany.	(FALSE)
INVERSE:	If I do not live in Albany, then I am not a New Yorker.	(FALSE)
CONTRAPOSITIVE:	If I am *not* a New Yorker, then I do not live in Albany.	(TRUE)

This example illustrates the truth value relationships between pairs of related conditional statements:

- A conditional statement and its converse may have the same or may have opposite truth values. In the previous example, the original statement was true, but its converse was false.
- A conditional statement and its contrapositive always agree in their truth values.
- The converse and inverse always have the same truth values.

MATH FACTS

Statements that always agree in their truth values are **logically equivalent**.

- A conditional statement and its contrapositive are logically equivalent, as are the converse and inverse.
- Starting with a true conditional statement, you can form another conditional that must be true by writing its contrapositive.

Example 1

Write the converse of each statement and indicate the truth values.

a. If two angles are right angles, then the angles are congruent.

b. If two lines are parallel, then the two lines do not intersect

Solution: The converse of a true statement may be true or may be false.

a.	ORIGINAL:	If two angles are right angles, then the angles are congruent."	(TRUE)
	CONVERSE:	If two angles are congruent, then the angles are right angles.	(FALSE)
b.	ORIGINAL:	If two lines are parallel, then the two lines do not intersect.	(TRUE)
	CONVERSE:	If two lines do not intersect, then the two lines are parallel.	(TRUE)

Example 2

Given the true statement "If I study, then I pass the test." Which statement must also be true?

(1) I study if I pass the test.
(2) If I do not study, then I do not pass the test.
(3) If I do not pass the test, then I did not study.
(4) If I pass the test, then I study.

Solution: Because it is given that the original statement is true, its contrapositive must also be true. Form the contrapositive by negating and then interchanging the If and Then statements:

ORIGINAL: If I study, then I pass the test.

CONTRAPOSITIVE: If I do *not* pass the test, then I did *not* study.

Look for the contrapositive among the answer choices.
The correct choice is **(3)**.

Role of Conditional Statements

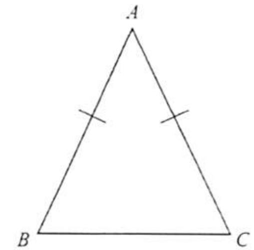
Theorems are often expressed in the form "If p, then q" where statement p is the *hypothesis* or what is given and statement q is the *conclusion* or what needs to be proved. A theorem that will eventually be proved is,

If two sides of a triangle are congruent (hypothesis or Given), **then** the angles opposite them are congruent (conclusion or Prove).

From this If–Then statement, you can determine the Given, Prove, and diagram:

If → Given: $\overline{AB} \cong \overline{AC}$.

Then → Prove: $\angle B \cong \angle C$.



After a proposed theorem is proved, forming related conditionals may suggest additional relationships that can be investigated. For instance, once the above theorem is proved, a likely question to ask is: If two *angles* of a triangle are congruent, are the sides opposite them congruent?

Biconditional Statements

The definition of a right angle can be written as the conditional,

ORIGINAL: If an angle is a right angle, then the angle measures 90°.

Since a definition is reversible, the definition could also be written as

CONVERSE: If an angle measures 90°, then the angle is a right angle.

As both forms of the definition are true, their conjunction is true:

CONJUNCTION: If an angle is a right angle, then the angle measures 90°
AND
If an angle measures 90°, then the angle is a right angle.

Combining a conditional statement and its converse in this way forms a *biconditional statement*. A **biconditional statement** is the conjuction of a conditional statement and its converse. A biconditional can be shortened by connecting the hypothesis and the conclusion of the original conditional statement with the phrase "if and only if" :

An angle is a right angle (Term being defined) *if and only if* the angle measures 90° (Distinguishing characteristic).

MATH FACTS

A **biconditional** statement has the general form,

$$\underbrace{\text{[Fact 1]}}_{\text{hypothesis}} \textbf{ if and only if } \underbrace{\text{[Fact 2]}}_{\text{conclusion}}$$

- When a definition is written as a biconditional, Fact 1 corresponds to the term that is being defined and Fact 2 is its distinguishing characteristic.
- The biconditional form of a definition emphasizes its reversibility. Most often, however, the two parts of a definition are connected by the word "is" simply because it is shorter.
- A biconditional is true only when both of its parts are either both true or both false. A proved theorem is expressed as a biconditional when its converse is also true.

Representing Conditional Statements Symbolically

A conditional statement can be represented symbolically as $p \to q$, read as "p implies q." The biconditional of p and q is denoted by $p \leftrightarrow q$ and read as "p if and only if q."

- The conditional $p \to q$ is always true except in the single instance when p is true and q is false.
- The biconditional $p \leftrightarrow q$ is true only when p and q are either both true or both false.

The accompanying table summarizes all the possible forms of a conditional statement.

Conditional	Sentence Form	Symbolic Form
Original	If p, then q.	$p \to q$
Converse	If q, then p.	$q \to p$
Inverse	If not p, then not q.	$\sim p \to \sim q$
Contrapositive	If not q, then not p.	$\sim q \to \sim p$
Biconditional	p if and only if q.	$p \leftrightarrow q$

logically equivalent

Check Your Understanding of Section 1.6

A. Multiple Choice.

1. What is the converse of the statement "If Alicia goes to Albany, then Ben goes to Buffalo"?
(1) If Alicia does not go to Albany, then Ben does not go to Buffalo.
(2) Alicia goes to Albany if and only if Ben goes to Buffalo.
(3) If Ben goes to Buffalo, then Alicia goes to Albany.
(4) If Ben does not go to Buffalo, then Alicia does not go to Albany.

2. Which statement is the inverse of "If the waves are small, I do not go surfing"?
(1) If the waves are not small, I do not go surfing.
(2) If I do not go surfing, the waves are small.
(3) If I go surfing, the waves are not small.
(4) If the waves are not small, I go surfing.

3. Which statement is logically equivalent to the statement "If you are an elephant, then you do not forget"?
(1) If you do not forget, then you are an elephant.
(2) If you do not forget, then you are not an elephant.
(3) If you are an elephant, then you forget.
(4) If you forget, then you are not an elephant.

4. Which statement is expressed as a biconditional?
(1) Two angles are congruent if they have the same measure.
(2) If two angles are both right angles, then they are congruent.
(3) Two angles are congruent if and only if they have the same measure.
(4) If two angles are congruent, then they are both right angles.

5. Let p represent "I like cake" and let q represent "I like ice cream." Which expression represents "If I do not like cake, then I do not like ice cream"?
(1) $\sim p \vee \sim q$ (2) $\sim p \wedge \sim q$ (3) $\sim p \to \sim q$ (4) $\sim(p \to q)$

6. Which statement is logically equivalent to "If I did not eat, then I am hungry"?
(1) If I am not hungry, then I did not eat.
(2) If I did not eat, then I am not hungry.
(3) If I am not hungry, then I did eat.
(4) If I am hungry, then I did eat.

7. Let p represent "$x > 5$" and let q represent "x is a multiple of 3." If $x = 12$, which statement is false?
(1) $p \leftrightarrow q$ (2) $p \wedge q$ (3) $\sim p \vee q$ (4) $\sim p \vee \sim q$

8. Let p represent "x is an even number greater than 10" and let q represent "x is *not* evenly divisible by 4." For which value of x is $p \wedge \sim q$ a true statement ?
(1) 8 (2) 14 (3) 18 (4) 20

9. Which statement is true when p is false and q is true?
(1) $p \leftrightarrow q$ (2) $p \wedge q$ (3) $\sim p \vee q$ (4) $\sim(p \vee q)$

10. Which statement is always true?
(1) $p \wedge \sim p$ (2) $p \vee \sim p$ (3) $p \rightarrow \sim p$ (4) $p \leftrightarrow \sim p$

11. Which statement is logically equivalent to "If it is Saturday, then I am not in school"?
(1) If I am not in school, then it is Saturday.
(2) If it is not Saturday, then I am in school.
(3) If I am in school, then it is not Saturday.
(4) If it is Saturday, then I am in school.

12. Which statement is the contrapositive of "If a triangle is a right triangle, then it has two complementary angles"?
(1) If a triangle is a right triangle, then it does not have two complementary angles.
(2) If a triangle does have two complementary angles, then it is not a right triangle.
(3) If a triangle is not a right triangle, then it has two complementary angles.
(4) If a triangle does not have two complementary angles, then it is a right triangle.

13. Which statement is the converse of "If the sum of two angles is 180°, then the angles are supplementary"?
(1) If two angles are supplementary, then their sum is 180°.
(2) If the sum of two angles is not 180°, then the angles are not supplementary.
(3) If two angles are not supplementary, then their sum is not 180°.
(4) If the sum of two angles is not 180°, then the angles are supplementary.

14. Which statement is expressed as a biconditional?
(1) Two angles are congruent if they have the same measure.
(2) If two angles are both right angles, then they are congruent.
(3) Two angles are congruent if and only if they have the same measure.
(4) If two angles are congruent, then they are both right angles.

15. What is the inverse of the statement "If Bob gets hurt, then the team loses the game"?
(1) If the team loses the game, then Bob gets hurt.
(2) Bob gets hurt if the team loses the game.
(3) If the team does not lose the game, then Bob does not get hurt.
(4) If Bob does not get hurt, then the team does not lose the game.

16. Given the true statement: "If a person is eligible to vote, then that person is a citizen." Which statement must also be true?
(1) Kayla is not a citizen; therefore, she is not eligible to vote.
(2) Juan is a citizen; therefore, he is eligible to vote.
(3) Marie is not eligible to vote; therefore, she is not a citizen.
(4) Morgan has never voted; therefore, he is not a citizen.

17. What is the converse of the statement "If the Sun rises in the east, then it sets in the west"?
(1) If the Sun does not set in the west, then it does not rise in the east.
(2) If the Sun does not rise in the east, then it does not set in the west.
(3) If the Sun sets in the west, then it rises in the east.
(4) If the Sun rises in the west, then it sets in the east.

18. Which statement cannot be written as a true biconditional?
(1) If the two lines are congruent, then they have the same measure.
(2) If two angles have the same measure, then they are congruent.
(3) If two angles are vertical angles, then they are congruent.
(4) If two lines are perpendicular, the lines intersect to form congruent adjacent angles.

B. *Show or explain how you arrived at your answer.*

19. Let r represent "You may vote in the general election" and let s represent "You are *at least* 18 years old." Using r and s, write in symbolic form: "You may *not* vote in the general election if and only if you are *less than* 18 years old."