CHAPTER 7

CIRCLES AND ANGLE MEASUREMENT

7.1 CIRCLE PARTS AND RELATIONSHIPS

KEY IDEAS

The two basic figures in the study of plane geometry are the triangle and the circle. A **circle** is a set of points in a plane at a fixed distance from a given point. The fixed distance is the **radius** of the circle and the given point is its **center**. A circle is named by its lettered center. If the center of a circle is point *O*, then that circle is referred to as circle *O*. A **semicircle** is one-half of a circle. **Congruent circles** are circles with congruent radii.

Points, Lines, and Segments of a Circle

If the distance from the center of a circle to a point *Y is less than* the radius, it is an **interior point** of the circle; if the distance from the center to point *X is greater than* the radius, then *X* is an **exterior point** of the circle.

A continuous set of points of the circle that trace out a curved portion of it is called an **arc**. An arc that is less than a semicircle is named by its two endpoints. The notation $\overset{\frown}{LM}$ refers to the *arc* whose endpoints are points *L* and *M*, as shown in Figure 7.1.

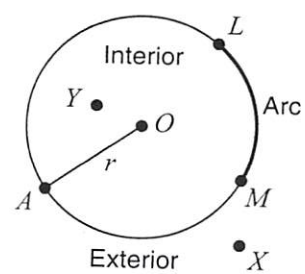
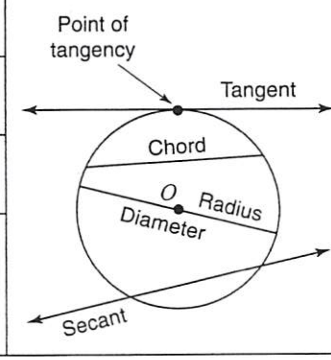


Figure 7.1 Interior and exterior points.

The accompanying table identifies some different types of lines and segments that contain points of a circle.

Special Line or Segment	Circle Diagram
A **chord** is a segment whose endpoints are points of the circle.	
A **diameter** is a chord that passes through the center of the circle.	
A **secant** is a line that intersects a circle at two points.	
A **tangent** is a line in the same plane as the circle that intersects it at exactly one point called the **point of tangency** or **point of contact.**	

Central Angles and Arcs

A **central angle** is an angle whose vertex is the center of a circle and whose sides contain radii of the circle. If radii $\overline{OA}$ and $\overline{OB}$ are the sides of a central angle that measures less than 180 degrees, as in Figure 7.2, then

- Points *A* and *B* and the points of the circle that are in the *interior* of the central angle form a **minor arc**.
- Points *A* and *B* and the points of the circle that are in the *exterior* of the central angle form a **major arc**.

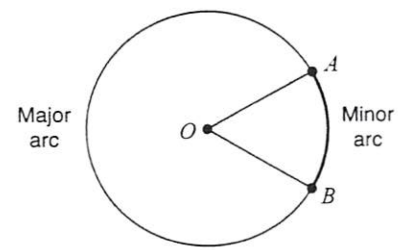


Figure 7.2 Classifying arcs.

If points *A* and *B* are the endpoints of a diameter, then each arc that is formed is a semicircle.

Naming and Measuring Arcs

To help distinguish minor arc *AB* from the major arc having the same two endpoints, a third letter is placed on the major arc. In Figure 7.3, $\overset{\frown}{APB}$ is the *major* arc with endpoints *A* and *B*.

- The **measure of a circle** is 360 degrees, and the **measure of a semi-circle** is 180 degrees.
- The **measure of minor arc** AB is the measure of central angle AOB that intercepts it. Thus,

$$m\widehat{AB} = m\angle AOB = 50.$$

- The **measure of major arc** APB is 360 minus the degree measure of its minor arc. Hence, $m\widehat{APB} = 360 - 50 = 310$.

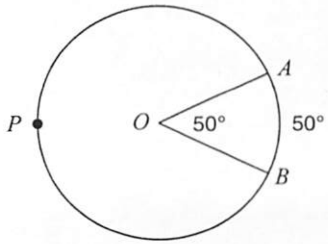


Figure 7.3 Degree Measure of Arcs.

Congruent Circles and Arcs

In the same or congruent circles, congruent central angles have **congruent arcs**, as shown in Figure 7.4. Although arcs AB and CD in Figure 7.5 have congruent central angles, their arcs are *not* congruent since they are in *concentric* rather than in congruent circles. **Concentric circles** are circles with the same center but unequal radii.

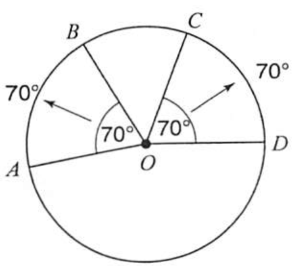


Figure 7.4 $\widehat{AB} \cong \widehat{CD}$.

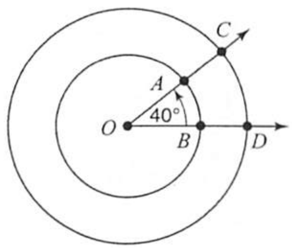


Figure 7.5 $\widehat{AB} \ncong \widehat{CD}$.

Congruent Arcs and Chords

In Figure 7.6, if chords $\overline{AB}$ and $\overline{CD}$ are congruent, then $\widehat{AB} \cong \widehat{CD}$. Conversely, if $\widehat{AB}$ and $\widehat{CD}$ are congruent, then $\overline{AB} \cong \overline{CD}$.

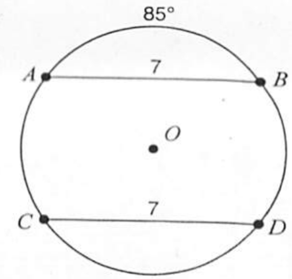


Figure 7.6 Congruent chords have congruent arcs.

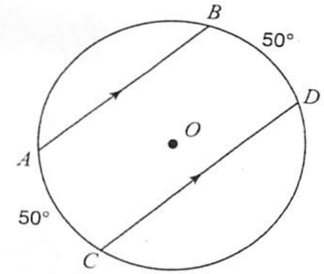


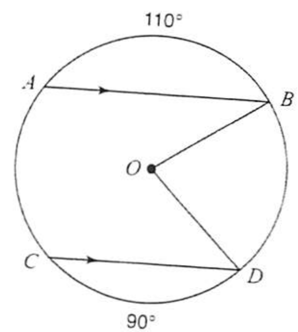
Figure 7.7 Parallel chords cut off congruent arcs.

In Figure 7.7, if chords $\overline{AB}$ and $\overline{CD}$ are parallel, then $\widehat{AC} \cong \widehat{BD}$.

> **Theorems: Arc–Chord Theorems**
> In the same or congruent circles:
> - If two chords are congruent, then their arcs are congruent.
> - If two arcs are congruent, then their chords are congruent.
> - If two chords are parallel, then their intercepted arcs are congruent.

Example 1

In the accompanying diagram, $\overline{AB}$ is parallel to $\overline{CD}$. If $m\widehat{AB} = 110$ and $m\widehat{CD} = 90$, what is the degree measure of central angle BOD?



Solution: Since parallel chords intercept equal arcs, let $x = m\widehat{BD} = m\widehat{AC}$. The sum of the degree measures of the arcs that comprise a circle is 360. Thus

$$\begin{aligned} m\widehat{AB} + m\widehat{BD} + m\widehat{CD} + m\widehat{AC} &= 360 \\ 110 + x \quad + 90 \quad + x &= 360 \\ 2x + 200 &= 360 \\ x = \frac{160}{2} &= 80 \end{aligned}$$

Hence, $m\widehat{BD} = 80$. Since a central angle and its intercepted arc have the same degree measure, $m\angle BOD = \mathbf{80}$.

Diameter Perpendicular to a Chord

When a diameter is perpendicular to a chord, the diameter bisects the chord and its minor and major arcs, as shown in Figure 7.8.

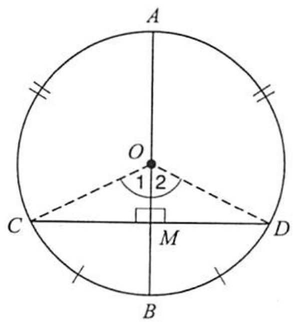


Figure 7.8 Diameter $\overline{AB} \perp$ chord $\overline{CD}$ makes $\overline{CM} \cong \overline{DM}$, $\overset{\frown}{CB} \cong \overset{\frown}{CD}$ and $\overset{\frown}{AC} \cong \overset{\frown}{AD}$.

> **Theorem: Diameter ⊥ Chord**
> If a diameter of a circle is perpendicular to a chord, then it bisects the chord and its arcs.

Paragraph Proof (Refer to Figure 7.8)

Given: Circle O, diameter $\overline{AB} \perp$ chord $\overline{CD}$.
Prove: a. $\overline{AB}$ bisects $\overline{CD}$.
b. $\overline{AB}$ bisects $\overset{\frown}{CD}$ and $\overset{\frown}{CAD}$.

a. Prove $\triangle COM \cong \triangle DOM$ to show that $\overline{CM} \cong \overline{DM}$.

- Draw radii $\overline{OC}$ and $\overline{OD}$ (two points determine a line).
- $\overline{OC} \cong \overline{OD}$ (Hyp) and $\overline{OM} \cong \overline{OM}$ (Leg). Right triangles COM and DOM are congruent by HL ≅ HL.
- $\overline{CM} \cong \overline{DM}$ (CPCTC) so $\overline{AB}$ bisects $\overline{CD}$.

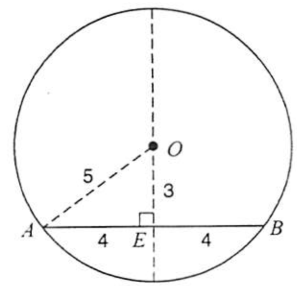
b. Use congruent central angles to obtain the required pairs of congruent arcs.

- Since $\angle 1 \cong \angle 2$ (CPCTC), $\overset{\frown}{CB} \cong \overset{\frown}{DB}$. Hence, $\overline{AB}$ bisects $\overset{\frown}{CD}$.
- $\angle AOC \cong \angle AOD$ since supplements of congruent angles are congruent.
- $\therefore \overset{\frown}{AC} \cong \overset{\frown}{AD}$ so $\overline{AB}$ bisects $\overset{\frown}{CAD}$.

Example 2

A chord is 3 inches from the center of a circle whose radius is 5 inches. What is the length of the chord?

Solution: In the accompanying diagram of circle O, $\overline{OA}$ is a radius, and OE is the distance of chord $\overline{AB}$ from center O. It is given that $OA = 5$ in and $OE = 3$ in. Because $\triangle OEA$ is a 3–**4**–5 right triangle, $AE = 4$. Since $\overline{OE}$ is perpendicular to chord $\overline{AB}$ and passes through center O, it lies on a diameter, so it bisects $\overline{AB}$. Hence $AE = BE = 4$ in.



The length of the chord $\overline{AB}$ is, therefore, 4 + 4 or **8 in.**

Equidistant Chords

In Figure 7.9, $\overline{AB} \cong \overline{CD}$. The lengths of perpendicular segments $\overline{OP}$ and $\overline{OQ}$ represent the distances of the segments from the center O. When a line through the center of a circle is perpendicular to a chord, it bisects the chord. Hence, $\overline{AP} \cong \overline{CQ}$, as halves of congruent segments are congruent. Since $\overline{OA} \cong \overline{OB}$, $\triangle OPA \cong \triangle OQC$, which means $\overline{OP} \cong \overline{OQ}$.

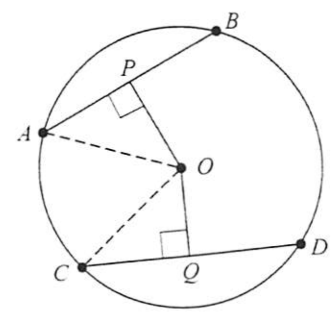


Figure 7.9 $\overline{AB} \cong \overline{CD} \Rightarrow OP = OQ$.

Conversely, if two chords are the same distance from the center of a circle, the chords are congruent.

> **Theorems: Equidistant Chord Theorems**
> In the same or congruent circles:
> - If two chords are congruent, they are the same distance from the center.
> - If two chords are the same distance from the center, they are congruent.

MATH FACTS

Deciding When Two Arcs Are Congruent

In the same circle or in congruent circles, two *arcs* are congruent when any one of the following statements is true:

- The central angles that intercept the arcs are congruent.
- The chords that the arcs determine are congruent.
- The arcs are between parallel chords.
- The arcs are formed by a diameter perpendicular to a chord.
- The arcs are semicircles.

Deciding When Two Chords Are Congruent

In the same circle or in congruent circles, two *chords* are congruent when

- The arcs that the chords intercept are congruent.
- The chords are the same distance from the center.

Check Your Understanding of Section 7.1

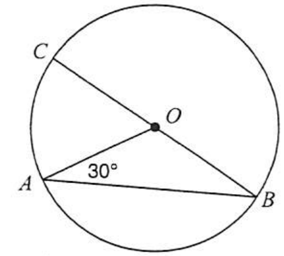
A. Multiple Choice

1. A chord 48 centimeters in length is 7 centimeters from the center of a circle. What is the number of centimeters in the length of a radius of this circle?
(1) 25 (2) 50 (3) 55 (4) $\sqrt{2,353}$

2. A chord is 5 inches from the center of a circle whose diameter is 26 inches. What is the number of centimeters in the length of the chord?
(1) 12 (2) 24 (3) 31 (4) 36

3. What is the distance, in inches, of a 30-inch chord from the center of a circle with a radius of 17 inches?
(1) 4 (2) 8 (3) 13 (4) 47

4. In the accompanying diagram of circle O, $\overline{BOC}$ is a diameter, radius $\overline{OA}$ is drawn, and $m\angle OAB = 30$. What is the measure of minor arc AC?
(1) 15 (3) 60
(2) 30 (4) 90



Exercise 4

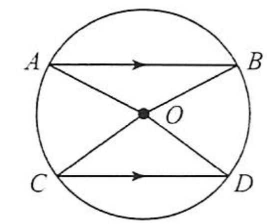
B. *Show or explain how you arrived at your answer.*

5–7. In the accompanying diagram, $\overline{AB}$ is parallel to $\overline{CD}$ in circle O.

5. If $m\widehat{BD} = 75$ and $m\widehat{CD} = 90$, find $m\angle AOB$.

6. If $m\angle BAO = 40$ and $m\widehat{CD} = 70$, find $m\widehat{BD}$.

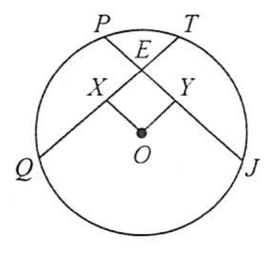
7. If $OC = CD$ and $m\widehat{AC} = 79$, find $m\angle AOB$.



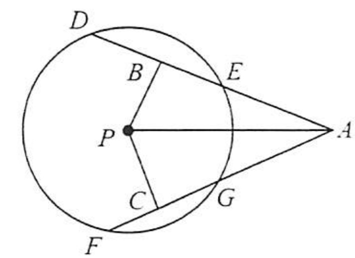
Exercises 5–7

8. In circle O, chord $\overline{AB}$ is 9 inches from the center. The diameter of the circle exceeds the length of $\overline{AB}$ by 2 inches. Find the length of $\overline{AB}$.

9. Given: In circle O, quadrilateral $OXEY$ is a square.
Prove: $\widehat{QP} \cong \widehat{JT}$.



Exercise 9



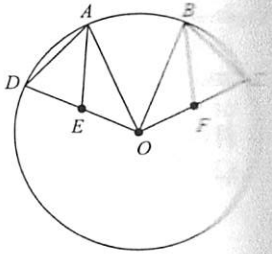
Exercise 10

10. Given: In circle P, $\overline{PB} \perp \overline{DE}$, $\overline{PC} \perp \overline{FG}$, $\overline{DE} \cong \overline{FG}$.
Prove: $\overline{PA}$ bisects $\angle FAD$.

11. In the accompanying diagram of circle O, points E and F are midpoints of radii $\overline{OD}$ and $\overline{OC}$, respectively, $\widehat{AD} \cong \widehat{BC}$.

Prove: a. $\triangle AOE \cong \triangle BOF$.

b. $\angle DAE \cong \angle CBF$



Exercise 11

7.2 TANGENTS AND CIRCLES

KEY IDEAS

Tangent segments drawn to a circle from the same exterior point are congruent. Circles may be tangent to each other. The same line may be tangent to more than one circle.

Properties of Tangents

In Figure 7.10, tangent line t intersects the circle at point A so radius $\overline{OA} \perp t$. Conversely, if line t is perpendicular to radius $\overline{OA}$ at A, then t is tangent to circle O at point A.

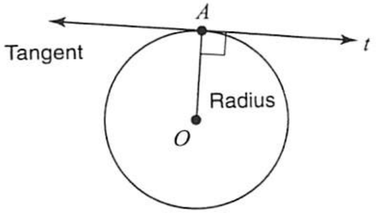


Figure 7.10 $\overline{OA} \perp t$.

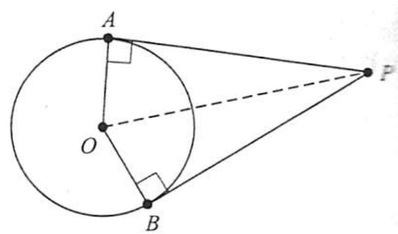


Figure 7.11 $\overline{PA} \cong \overline{PB}$.

In Figure 7.11, tangent segments $\overline{PA}$ and $\overline{PB}$ are drawn. Since $\triangle OAP \cong \triangle OBP$ by Hy–Leg, $\overline{PA} \cong \overline{PB}$.

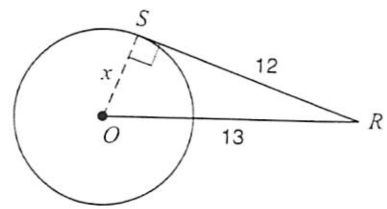
Theorems: Tangent Theorems

- **Theorem:** If a line is tangent to a circle, then it is perpendicular to the radius drawn to the point of contact.
- **Theorem:** If a line is perpendicular to a radius at its endpoint on a circle, then the line is tangent to the circle at that point.
- **Theorem:** If two tangent segments are drawn to a circle from the same exterior point, then they are congruent.

Example 1

From a point R that is 13 inches from the center of circle O, $\overline{RS}$ is drawn tangent to circle O at point S. If $RS = 12$ inches, what is the area of circle O expressed in terms of π?

Solution: Draw radius $\overline{OS}$. Since $\overline{OS} \perp \overline{RS}$, $\triangle OSR$ is a right triangle whose side lengths form a **5–12–13** Pythagorean triple where $x = 5$. The area, A, of a circle is given by the formula $A = \pi r^2$. Because $r = 5$ inches, $A = \pi \times 5^2 = 25\pi$ in^2.

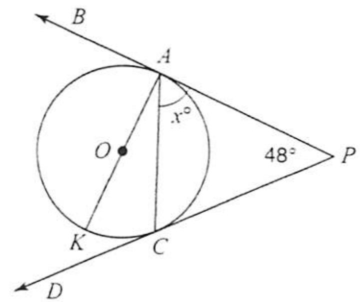


Example 2

In the accompanying figure, $\overleftrightarrow{PAB}$ and $\overleftrightarrow{PCD}$ are tangent to circle O at points A and B, respectively, m$\angle P = 48$, chord $\overline{AC}$ and diameter $\overline{AOK}$ are drawn. Find m$\angle KAC$.

Solution: Tangents $\overline{PA}$ and $\overline{PC}$ are congruent, so in $\triangle APC$, m$\angle PAC$ = m$\angle PCA = x$:

$$x + x + 48 = 180$$
$$2x = 180 - 48$$
$$\frac{2x}{2} = \frac{132}{2}$$
$$x = 66$$



Angle KAP is a right angle so m$\angle KAP = 90$.

$$m\angle KAC + m\angle PAC = m\angle KAP$$
$$m\angle KAC + 66 = 90$$
$$m\angle KAC = 90 - 66$$
$$m\angle KAC = \mathbf{24}$$

Tangent Circles

Tangent circles are circles in the same plane that are tangent to the same line at the same point. Circles may be tangent internally or externally, as shown in Figure 7.12.

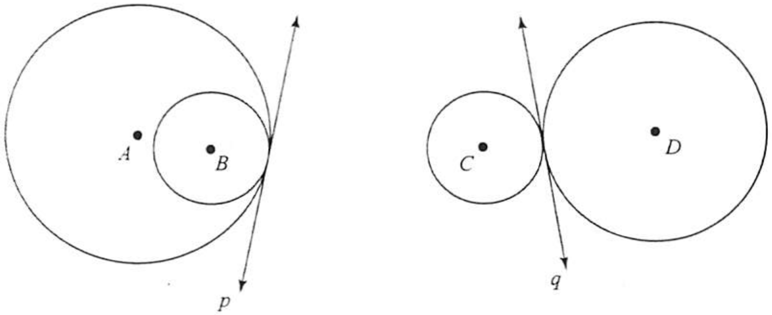


Figure 7.12 Tangent circles.

- If the circles lie on the same side of the common tangent line, the circles are **tangent internally.**
- If the circles lie on opposite sides of the common tangent line, the circles are **tangent externally**.

Common Tangents

A line that is tangent to two circles is a **common tangent**. Common tangents may be *external or internal*. A common *external* tangent does *not* intersect the line through the centers of the two circles at a point between the two circles. A common *internal* tangent intersects the line joining the centers of the two circles at a point between the two circles. Figure 7.13 shows the possible numbers of common tangents drawn to two circles.

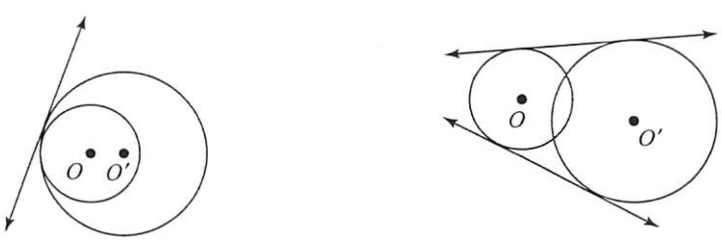


Figure 7.14 Common tangents.

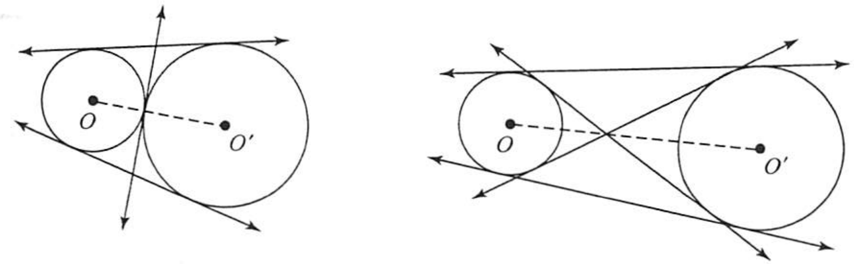


Figure 7.14 (continued) Common tangents.

The **length of a common tangent** is the length of the segment whose endpoints are the two points of contact.

Circumscribed and Inscribed Polygons

A polygon is **inscribed in a circle** if all the vertices of the polygon are points on the circle, as in Figure 7.14. The circle is said to be **circumscribed about the polygon**.

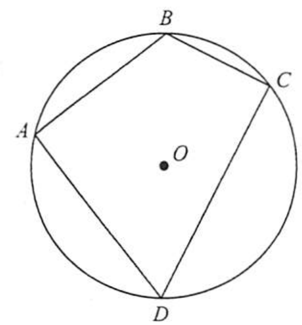


Figure 7.14 Quadrilateral *ABCD* is inscribed in circle *O*.

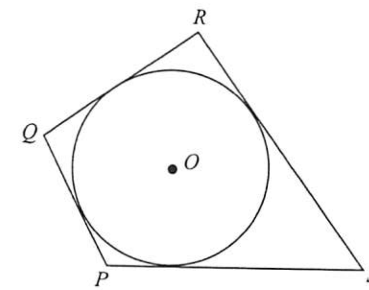


Figure 7.15 Quadrilateral *PQRS* is circumscribed about circle *O*.

A polygon is **circumscribed about a circle** if each side of the polygon is tangent to the circle, as in Figure 7.15. The circle is said to be **inscribed in the polygon**.

The **center** of a regular polygon is the point in the interior of the polygon that is equidistant from the vertices of the polygon. The centers of the inscribed and circumscribed circles of a regular polygon coincide with the center of the regular polygon.

Check Your Understanding of Section 7.2

A. Multiple Choice

1. If two circles are internally tangent, what is the total number of common tangents that can be drawn to the circles?
(1) 1 (2) 2 (3) 3 (4) 0

2. From exterior point L of circle O, tangent segments $\overline{LP}$ and $\overline{LQ}$ are drawn such that m$\angle PLQ = 60$. If a radius of circle O measures 6, what is the distance from the center of the circle to point L?
(1) $3\sqrt{3}$ (2) 2 (3) 3 (4) 12

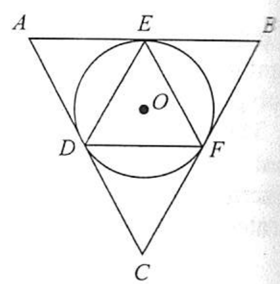
3. If two circles with diameters of 6 and 2 are internally tangent, then the distance between their centers is
(1) 0 (2) 2 (3) 3 (4) 4

4. The radii of two circles are r and R, and the distance between their centers is d. If $d = r + R$, then the number of common tangents the two circles have is
(1) 1 (2) 2 (3) 3 (4) 4

5. If circles O and O' do not intersect, the maximum number of common tangents they may have is
(1) 1 (2) 2 (3) 3 (4) 4

6. If for two given circles exactly two common tangents are possible, the circles
(1) intersect at two points
(2) are concentric
(3) are tangent internally
(4) are tangent externally

7. In the accompanying diagram, equilateral $\triangle ABC$ is circumscribed about circle O, and equilateral $\triangle DEF$ is inscribed in the same circle. What is the ratio of AB to DE?
(1) 4:1 (3) $2\sqrt{3}:1$
(2) 2:1 (4) $\sqrt{3}:1$

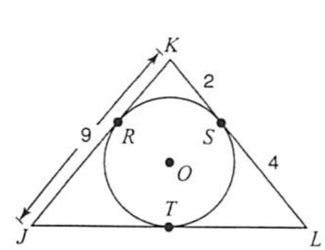


8. Tangents $\overrightarrow{PA}$ and $\overrightarrow{PB}$ are tangent to circle O at points A and B, respectively, and radii $\overline{OA}$ and $\overline{OB}$ are drawn. Which statement is always true?
(1) m$\angle P$ = m$\angle AOB$ (3) m$\angle P$ + m$\angle AOB = 90$
(2) m$\angle P$ > m$\angle AOB$ (4) m$\angle P$ + m$\angle AOB = 180$

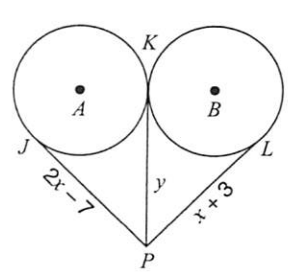
B. Show or explain how you arrived at your answer.

9. Externally tangent circles A and B have diameters of 8 inches and 4 inches, respectively. Point P lies on their common internal tangent at a distance of 3 inches from the point at which the two circles touch. Find the perimeter of $\triangle APB$ correct to the *nearest tenth of an inch*.

10. In the accompanying figure, $\overline{JK}$, $\overline{KL}$, and $\overline{JL}$ are tangent to circle O at points R, S, and T, respectively. What is the length of $\overline{JTL}$?



Exercise 10

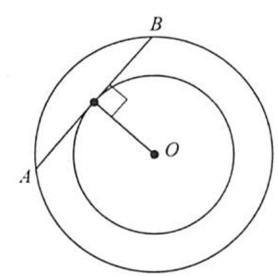


Exercise 11

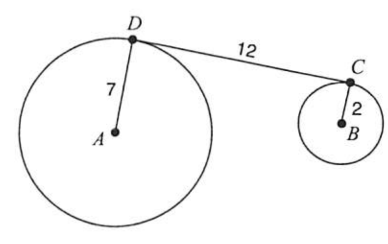
11. In the accompanying figure, $\overline{PJ}$ and $\overline{PL}$ are tangent segments drawn to circles A and B, respectively, and $\overline{PK}$ is tangent to both circles at K. Find the values of x and y.

12. From a point R that is 15 inches from the center of circle O, $\overline{RA}$ is drawn tangent to circle O at point A. If the area of circle O is 64π, what is the length of $\overline{RA}$?

13. In the accompanying diagram of circle O, the lengths of the radii of the two concentric circles are 4 and 5. Chord $\overline{AB}$ of the larger circle is tangent to the smaller circle. What is the length of $\overline{AB}$?

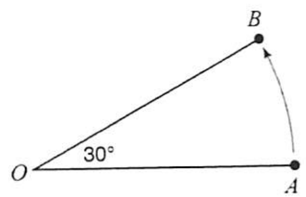


Exercise 13



Exercise 14

5. The area of a circle is 16π. What is the length of a side of the regular hexagon inscribed in the circle?
(1) 8 (2) 2 (3) 6 (4) 4



6. In the accompanying figure, a radar tracking beam is centered at O and sweeps in a circular pattern an angle of 30° as the end of the beam moves from point A to point B in the counterclockwise direction. If the length of $\overset{\frown}{AB}$ of circle O is 6π kilometers, what is the number of square kilometers in the area of the region that the radar beam covers when it moves from A to B?
(1) 36π (2) 54π (3) 72π (4) 108π

7. Kira buys a large circular pizza that is divided into eight equal slices. She measures along the outer edge of the crust of one piece and finds the distance to be $5\frac{1}{2}$ inches. What is the *diameter* of the pizza to the *nearest inch*?
(1) 14 (2) 8 (3) 7 (4) 15

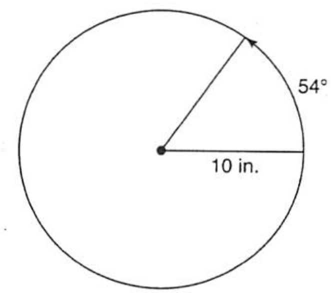
8. In a circle, the minor sector formed by two perpendicular radii has an area of 16π. The length of the *major* intercepted arc is
(1) 12π (2) 8π (3) 6π (4) 4π

9. A regular hexagon is inscribed in a circle. What is the ratio of the length of a side of the hexagon to the minor arc that it intercepts?
(1) $\frac{\pi}{6}$ (2) $\frac{3}{6}$ (3) $\frac{3}{\pi}$ (4) $\frac{6}{\pi}$

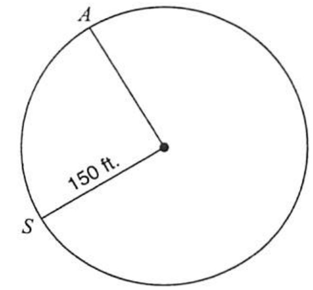
B. *Show or explain how you arrived at your answer.*

10. In circle O, sector AOB has a central angle of 40°. If the area of the sector is 4π square inches, find in terms of π the number of inches in the length of minor arc AB.

11. A ball is rolling in a circular path that has a radius of 10 inches, as shown in the accompanying diagram. What distance has the ball rolled when the subtended arc measures 54°? Express your answer to the *nearest hundredth of an inch.*



Exercise 11



Exercise 12

12. Kathy and Tami are at point A on a circular track that has a radius of 150 feet, as shown in the accompanying diagram. They run counterclockwise along the track from A to S, a distance of 247 feet. Find, to the *nearest degree,* the measure of minor arc AS.

13. A circle is circumscribed about a traffic sign that has the shape of a regular octagon. If the area of the circle is 256π in^2, find to the *nearest tenth* of an inch the distance along the circle, between two consecutive vertices of the octagon?

14. If each side of a regular polygon with 12 sides measures 54.0 cm, find
a. The length of the minor arc intercepted by a side of the polygon correct to the *nearest tenth* of a centimeter
b. The area of the polygon correct to the *nearest square centimeter*

7.4 CIRCLE ANGLE MEASUREMENT

KEY IDEAS

The vertex of an angle whose sides are chords, secants, or tangents may lie on the circle, inside the circle, or outside the circle. In each case, the location of the vertex determines the relationship between the measure of the angle and the measures of the intercepted arc or arcs.

Vertex on the Circle: Inscribed Angle

An **inscribed angle** of a circle is an angle whose vertex is on the circle and whose sides are chords. In Figure 7.18, inscribed angle ABC intercepts $\overset{\frown}{AC}$. An inscribed angle is measured by one-half of the measure of its intercepted arc:

$$m\angle ABC = \frac{1}{2} m\overset{\frown}{AC} = \frac{1}{2}(50) = 25$$

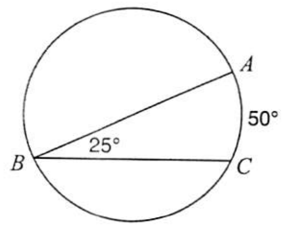


Figure 7.18 Measuring an inscribed angle.

Vertex on the Circle: Chord–Tangent Angle

An angle whose vertex is on the circle and whose sides are a tangent and a chord is measured by one-half of the measure of its intercepted arc. In Figure 7.19:

$$m\angle RST = \frac{1}{2} m\overset{\frown}{RS} = \frac{1}{2}(140) = 70$$

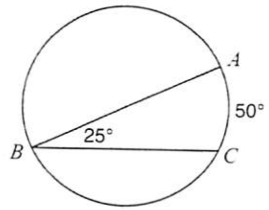
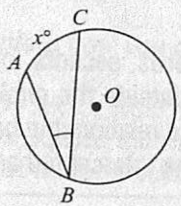
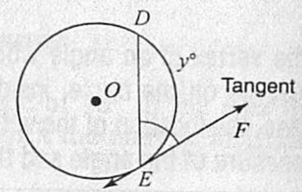


Figure 7.19 Measuring an angle formed by a tangent and a chord.

> **Theorem: Inscribed Angle Theorem**
> The measure of an inscribed angle is equal to one-half of the measure of its intercepted arc.



$$m\angle ABC = \frac{1}{2}x$$

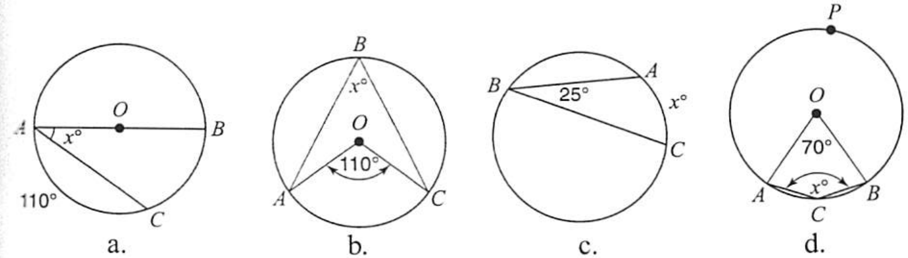


$$m\angle DEF = \frac{1}{2}y$$

> **Theorem: Chord–Tangent Theorem**
> The measure of an angle formed by a tangent and a chord drawn to the point of tangency is one-half of the measure of its intercepted arc.

Example 1

In each case, find the value of x.



Solution:

a. $\overset{\frown}{ABC}$ is a semicircle, so $m\overset{\frown}{BC} = 180 - 110 = 70$.

$$x = \frac{1}{2} m\overset{\frown}{BC} = \frac{1}{2}(70) = \mathbf{35}$$

b. $\angle AOC$ is a central angle, so $m\overset{\frown}{AC} = 110$.

$$m\angle ABC = \frac{1}{2} m\overset{\frown}{AC} = \frac{1}{2}(110) = \mathbf{55}$$

c. The degree measure of an arc intercepted by an inscribed angle must be twice the degree measure of the inscribed angle. Hence

$$x = 2(m\angle ABC) = 2(25) = \mathbf{50}$$

d. $\angle AOB$ is a central angle, so $m\widehat{ACB} = 70$.

$$m\widehat{APB} = 360 - 70 = 290$$

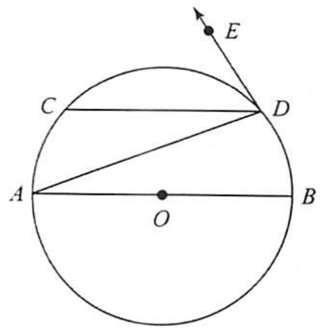
$$x = \frac{1}{2}m\widehat{APB}$$

$$= \frac{1}{2}(290)$$

$$= \mathbf{145}$$

Example 2

In the accompanying diagram, $\overrightarrow{DE}$ is tangent to circle O at D, $\overline{AOB}$ is a diameter, and $\overline{CD}$ is parallel to $\overline{AOB}$. If $m\angle DAB = 21$, find $m\angle CDE$.



Solution:

- Angle CDE intercepts arc CD, so the degree measure of this arc is needed. Since the sum of the degree measures of the arcs of a semicircle is 180:

$$m\widehat{AC} + m\widehat{CD} + m\widehat{DB} = 180.$$

- The degree measure of inscribed angle DAB is 21. The degree measure of its intercepted arc, $\widehat{DB}$, must be twice as great, or 42. Since parallel chords intercept equal arcs, $m\widehat{AC} = m\widehat{DB} = 42$. Hence:

$$42 + m\widehat{CD} + 42 = 180$$

$$m\widehat{CD} = 180 - 84$$

$$= 96$$

- Since $\angle CDE$ is formed by a chord and a tangent:

$$m\angle CDE = \frac{1}{2}m\widehat{CD}$$

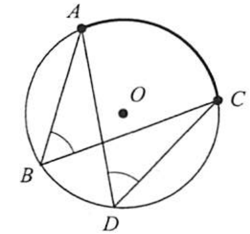
$$= \frac{1}{2}(96)$$

$$= \mathbf{48}$$

Corollaries of Inscribed Angle Theorem

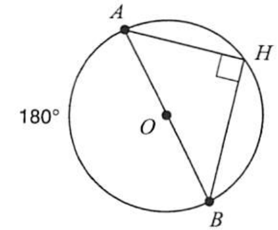
Several important relationships follow directly from the Inscribed Angle Theorem.

- Corollary 1: If two inscribed angles intercept the same or congruent arcs, then the angles are congruent. In the accompanying figure, inscribed angles B and D are congruent because they intercept the same arc, $\widehat{AC}$.



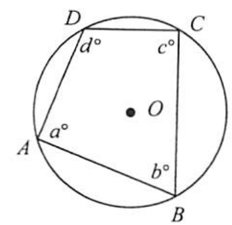
- Corollary 2: An angle inscribed in a semicircle is a right angle. In the accompanying figure, because $\angle AHB$ is inscribed in a semicircle, its intercepted arc is also a semicircle so

$$m\angle AHB = \frac{1}{2}m\widehat{AB} = \frac{1}{2}(180) = 90$$



- Corollary 3: The opposite angles of an inscribed quadrilateral are supplementary. In the accompanying figure,

$$a + c = 180 \quad \text{and} \quad b + d = 180$$



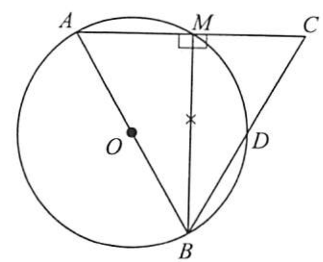
Circle Proofs

When the sides of triangles cut off arcs of a circle, circle angle–measurement relationships can sometimes be used to prove that pairs of angles in these triangles are congruent.

Example 3

Given: In circle O, $\overline{AB}$ is a diameter, $\widehat{AM} \cong \widehat{DM}$.

Prove: $\triangle AMB \cong \triangle CMB$.



Solution: See the proof.

PLAN: The triangles can be proved congruent by using the ASA postulate.

Proof

Statement		Reason
1. In circle O, $\overline{AB}$ is a diameter.		1. Given.
2. $\angle AMB$ is a right angle.		2. An angle inscribed in a semicircle is a right angle.
3. $\angle CMB$ is a right angle.		3. The supplement of a right angle is a right angle.
4. $\angle AMB \cong \angle CMB$	Angle	4. All right angles are congruent.
5. $\overline{MB} \cong \overline{MB}$	Side	5. Reflexive property.
6. $\overparen{AM} \cong \overparen{DM}$		6. Given.
7. $\angle ABM \cong \angle CBM$	Angle	7. Inscribed angles that intercept congruent arcs are congruent.
8. $\triangle AMB \cong \triangle CMB$		8. ASA postulate.

Vertex of Angle: Inside the Circle

When two chords intersect in the interior of a circle, as in Figure 7.20, each of the angles formed are opposite two arcs of the circle. The angle formed by the two chords is measured by one-half the *sum* of the measures of the two intercepted arcs. Thus

$$m\angle AEC = \frac{1}{2}(m\overparen{AC} + m\overparen{BD})$$

$$= \frac{1}{2}(150 + 70)$$

$$= 110$$

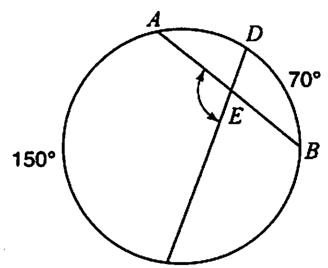
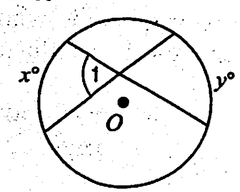


Figure 7.20 Vertex of angle inside circle.

Theorem: Chord–Chord Angle Theorem

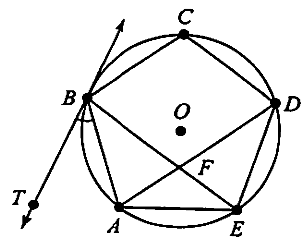
The measure of an angle formed by two chords intersecting inside a circle is equal to one-half of the sum of the measures of its two intercepted arcs.



$$m\angle 1 = \frac{1}{2}(x + y)$$

Example 4

In the accompanying diagram, regular pentagon $ABCDE$ is inscribed in circle O. Chords $\overline{AD}$ and $\overline{BE}$ intersect at F, and $\overrightarrow{BT}$ is tangent to circle O at B. Find:



a. $m\angle ABT$ b. $m\angle AFE$

Solution:

a. In a regular pentagon all five sides have the same length; therefore, circle O is divided into five equal arcs. The degree measure of each arc is, therefore, $\frac{360}{5}$ or 72. Hence:

$$m\angle ABT = \frac{1}{2}m\overparen{AB} = \frac{1}{2}(72) = \mathbf{36}$$

b. Angle AFE intercepts arcs BCD and AE. Since $m\overparen{AE} = 72$ and $m\overparen{BCD} = m\overparen{BC} + m\overparen{CD} = 72 + 72 = 144$:

$$m\angle AFE = \frac{1}{2}(\overparen{BCD} + m\overparen{AE})$$

$$= \frac{1}{2}(144 + 72)$$

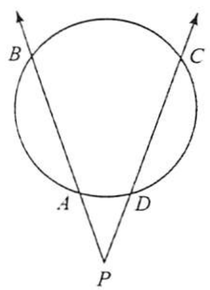
$$= \frac{1}{2}(216)$$

$$= \mathbf{108}$$

Vertex of Angle: Outside the Circle

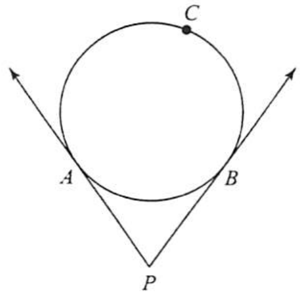
The vertex of an angle whose sides intercept arcs of a circle may lie in the exterior of a circle, as illustrated in Figure 7.21.

Secant–Secant Angle



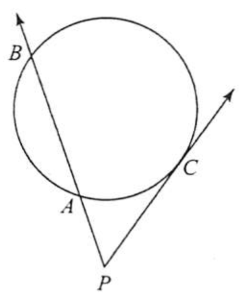
$$m\angle P = \frac{1}{2}(m\overparen{BC} - m\overparen{AD})$$

Tangent–Tangent Angle



$$m\angle P = \frac{1}{2}(m\overparen{ACB} - m\overparen{AB})$$

Tangent–Secant Angle



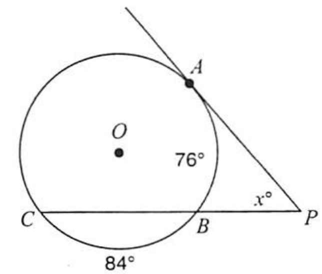
$$m\angle P = \frac{1}{2}(m\overparen{BC} - m\overparen{AC})$$

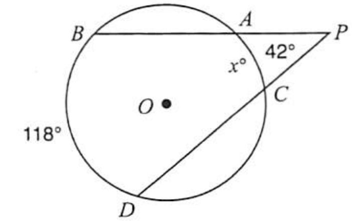
Figure 7.21 Angles whose vertices are in the exterior of the circle.

> **Theorems: Sec–Sec, Tan–Tan, Tan–Sec Angle Theorem**
> The measure of an angle formed by two secants, two tangents, or a tangent and a secant intersecting in the exterior of a circle is one-half of the difference in the measures of its two intercepted arcs.

Example 5

In each case, find the value of x.





Solution:

a.
$$m\overparen{AC} + 76 + 84 = 360$$
$$m\overparen{AC} = 360 - 160$$
$$= 200$$

Hence:

$$x = \frac{1}{2}(m\overparen{AC} - m\overparen{AB})$$
$$= \frac{1}{2}(200 - 76)$$
$$= \frac{1}{2}(124)$$
$$= \mathbf{62}$$

b.
$$m\angle P = \frac{1}{2}(m\overparen{BD} - m\overparen{AC})$$
$$42 = \frac{1}{2}(118 - x)$$
$$84 = 118 - x$$
$$x = 118 - 84$$
$$= \mathbf{34}$$

Example 6

In the accompanying diagram, $\overrightarrow{PA}$ is tangent to circle O at point A. Secant $\overline{PBC}$ is drawn. Chords $\overline{CA}$ and $\overline{BD}$ intersect at point E. If

$$m\overparen{AD} : m\overparen{AB} : m\overparen{DC} : m\overparen{BC} = 2:3:4:6$$

find the degree measure of each of the numbered angles.

Solution: First find the degree measures of the arcs of the circle. Let $2x = m\overparen{AD}$. Then

$$3x = m\overparen{AB}, \quad 4x = m\overparen{CD}, \quad 6x = m\overparen{BC}.$$

Since the sum of the degree measures of the arcs of a circle is 360:

$$\begin{aligned} m\overset{\frown}{AD} + m\overset{\frown}{AB} + m\overset{\frown}{DC} + m\overset{\frown}{BC} &= 360 \\ 2x + 3x + 4x + 6x &= 360 \\ 15x &= 360 \\ x = \frac{360}{15} &= 24 \end{aligned}$$

Hence

$$\begin{aligned} m\overset{\frown}{AD} &= 2x = 2(24) = 48 \\ m\overset{\frown}{AB} &= 3x = 3(24) = 72 \\ m\overset{\frown}{DC} &= 4x = 4(24) = 96 \\ m\overset{\frown}{BC} &= 6x = 6(24) = 144 \end{aligned}$$

- *∠1 is an inscribed angle:*

$$\begin{aligned} m\angle 1 &= \frac{1}{2} m\overset{\frown}{AB} \\ &= \frac{1}{2}(72) \\ &= \mathbf{36} \end{aligned}$$

- *∠2 is a chord–chord angle:*

$$\begin{aligned} m\angle 2 &= \frac{1}{2}(m\overset{\frown}{AD} + m\overset{\frown}{BC}) \\ &= \frac{1}{2}(48 + 144) \\ &= \mathbf{96} \end{aligned}$$

- *∠3 is a tangent–chord angle:*

$$\begin{aligned} m\angle 3 = \frac{1}{2} m\overset{\frown}{AC} &= \frac{1}{2}(m\overset{\frown}{AD} + m\overset{\frown}{DC}) \\ &= \frac{1}{2}(48 + 96) \\ &= \mathbf{72} \end{aligned}$$

- *∠4 is a secant–tangent angle:*

$$\begin{aligned} m\angle 4 &= \frac{1}{2}(m\overset{\frown}{AC} - m\overset{\frown}{AB}) \\ &= \frac{1}{2}(144 - 72) \\ &= \mathbf{36} \end{aligned}$$

- ∠5: Angles 5 and *CBD* are supplementary. The measure of ∠5 may be found indirectly by first finding m∠*CBD*.

$$\begin{aligned} m\angle CBD &= \frac{1}{2} m\overset{\frown}{DC} = \frac{1}{2}(96) = 48 \\ m\angle 5 &= 180 - m\angle CBD \\ &= 180 - 48 \\ &= \mathbf{132} \end{aligned}$$

MATH FACTS

The measure of an angle whose sides cut off arcs of a circle depends on the location of the vertex of the angle.

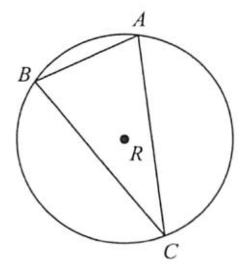
Position of Vertex of Angle	Measure of Angle
• On the circle	• $\frac{1}{2}$ the degree measure of the intercepted arc.
• Interior of the circle	• $\frac{1}{2}$ the *sum* of the degree measures of the two intercepted (opposite) arcs.
• Exterior of the circle	• $\frac{1}{2}$ the *difference* of the degree measures of the two intercepted arcs.

Check Your Understanding of Section 7.4

A. *Multiple Choice*

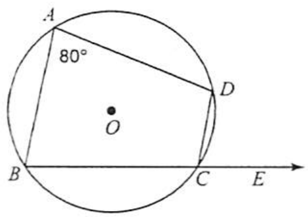
1. Tangent $\overline{PA}$ and secant $\overline{PBC}$ are drawn to circle *O* from external point *P*, and chord $\overline{AB}$ is drawn. If $m\overset{\frown}{AC} = 2\, m\overset{\frown}{AB}$, what is the ratio of m∠*PAB* to m∠*ABC*?
(1) 1:1 (2) 1:2 (3) 3:2 (4) 1:4

2. In the accompanying diagram, △*ABC* is inscribed in circle *R*. If $m\overset{\frown}{AB} = 2x$, $m\overset{\frown}{BC} = 5x$, and $m\overset{\frown}{AC} = 150$, which type of triangle is △*ABC*?
(1) right
(2) isosceles
(3) equilateral
(4) obtuse

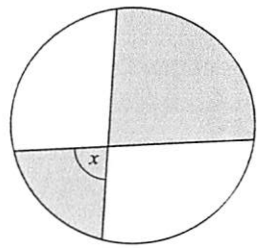


3. In circle O, $\overline{PA}$ and $\overline{PB}$ are tangent to the circle from point P. If the ratio of the measure of major arc AB to the measure of minor arc AB is 5:1, then m$\angle P$ is
(1) 60 (2) 90 (3) 120 (4) 180

4. In the accompanying diagram, quadrilateral $ABCD$ is inscribed in circle O, m$\angle BAD$ = 80 and $\overline{BCE}$ is drawn. What is m$\angle DCE$?
(1) 60 (2) 80 (3) 100 (4) 120



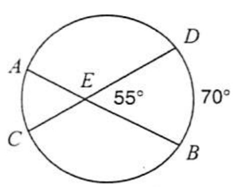
Exercise 4



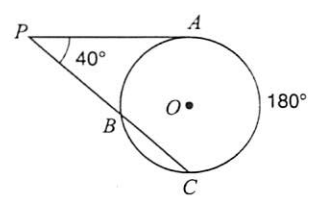
Exercise 5

5. The accompanying diagram shows a child's spin toy that is constructed from two chords intersecting in a circle. The curved edge of the larger shaded section is one-quarter of the circumference of the circle, and the curved edge of the smaller shaded section is one-fifth of the circumference of the circle. What is the measure of angle x?
(1) 40 (2) 72 (3) 81 (4) 108

6. In the accompanying diagram, chords $\overline{AB}$ and $\overline{CD}$ intersect at E. If m$\angle DEB$ = 55 and m$\widehat{DB}$ = 70, what is m$\widehat{AC}$?
(1) 15 (2) 40 (3) 55 (4) 70



Exercise 6

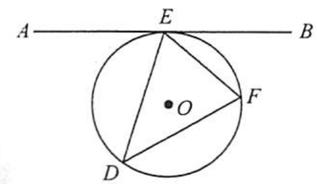


Exercise 7

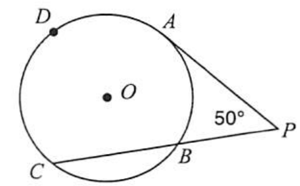
7. In the accompanying diagram of circle O, the measure of angle P formed by tangent $\overline{PA}$ and secant $\overline{PBC}$ is 40. If m$\widehat{AC}$ = 180, find m$\widehat{AB}$.
(1) 50 (2) 70 (3) 100 (4) 140

8. Two tangents to a circle from the same external point intercept a major arc whose measure is 210. What is the measure of the angle formed by the tangents?
(1) 30 (2) 75 (3) 90 (4) 150

9. In the accompanying figure, $\overline{AB}$ is tangent to circle O at E, $\overline{DE}$, $\overline{EF}$, and $\overline{FD}$ are chords, and m$\angle AEF$ = 136. What is m$\angle D$?
(1) 68 (2) 22 (3) 88 (4) 44



Exercise 9



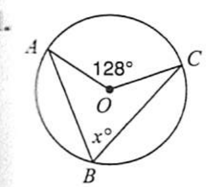
Exercise 10

10. In the accompanying diagram, tangent $\overline{PA}$ and secant $\overline{PBC}$ are drawn to circle O. If m$\widehat{ADC}$ is twice m$\widehat{AB}$ and m$\angle P$ is 50, what is m$\widehat{AB}$?
(1) 25 (2) 50 (3) 100 (4) 200

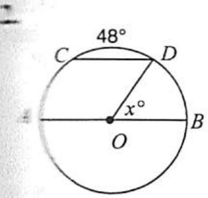
B. *Show or explain how you arrived at your answer.*

11–14. Find the value of x.

11.

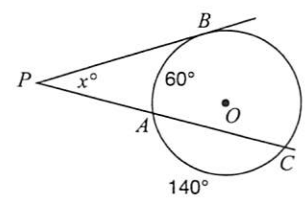


12.

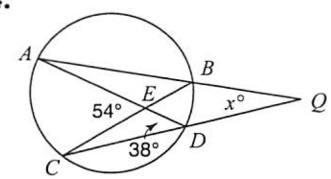


$\overline{AB} \parallel \overline{CD}$

13.

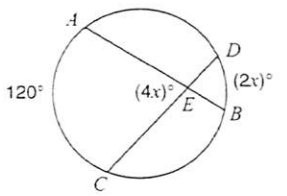


14.

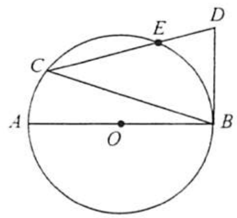


15. Point P lies outside circle O, which has a diameter of $\overline{AOC}$. The angle formed by tangent $\overline{PA}$ and secant $\overline{PBC}$ measures 30°. Find the number of degrees in the measure of minor arc CB.

16. In the accompanying diagram, chords $\overline{AB}$ and $\overline{CD}$ intersect at E. If $m\angle AEC = 4x$, $m\overset{\frown}{AC} = 120$, and $m\overset{\frown}{DB} = 2x$, what is the value of x?



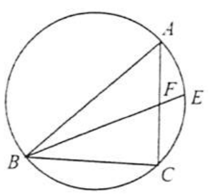
Exercise 16



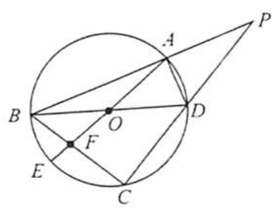
Exercise 17

17. In the accompanying diagram, $\overline{CED}$ is a secant, $\overline{BD}$ is tangent to circle O at B, $\overline{BC}$ is a chord, and $\overline{BOA}$ is a diameter. If $m\overset{\frown}{AC} : m\overset{\frown}{CB} = 1:4$ and $m\overset{\frown}{CE} = 68$, find $m\angle BDC$.

18. In the accompanying diagram, chord $\overline{BE}$ bisects $\angle ABC$. If $m\angle ABC = 70$ and $m\overset{\frown}{BAE} = 200$, find $m\angle AFE$.



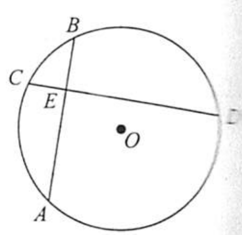
Exercise 18



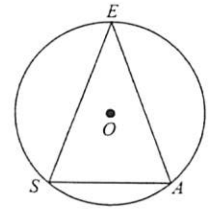
Exercise 19

19. In the accompanying diagram of circle O, diameters $\overline{BD}$ and $\overline{AE}$, secants $\overline{PAB}$ and $\overline{PDC}$, and chords $\overline{AD}$ and $\overline{BC}$ are drawn. Diameter $\overline{AE}$ intersects chord $\overline{BC}$ at F. If $m\angle ABD = 20$ and $m\angle BFE = 75$, find $m\angle P$.

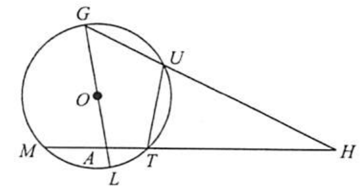
20. In the accompanying diagram of circle O, chords $\overline{AB}$ and $\overline{CD}$ intersect at E and $m\overset{\frown}{AC} : m\overset{\frown}{CB} : m\overset{\frown}{BD} : m\overset{\frown}{DA} = 4:2:6:8$. What is $m\angle DEB$?



21. A machine part consists of a circular wheel with an inscribed triangular plate, as shown in the accompanying diagram. If $\overline{SE} \cong \overline{EA}$, $SE = 10$ inches, and $m\overset{\frown}{SE} = 140$, find the length of $\overline{SA}$ to the *nearest tenth of an inch*.



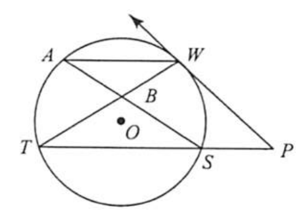
Exercise 21



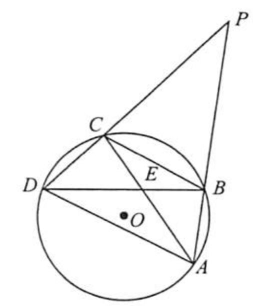
Exercise 22

22. Given circle O with diameter $\overline{GOAL}$; secants $\overline{HUG}$ and $\overline{HTAM}$ intersect at point H; $m\overset{\frown}{GM} : m\overset{\frown}{ML} : m\overset{\frown}{LT} = 7:3:2$; and $\overline{GU} \cong \overline{UT}$. Find the ratio of $m\angle UGL$ to $m\angle H$.

23. In circle O, tangent $\overrightarrow{PW}$ and $\overline{PST}$ are drawn. Chord $\overline{WA}$ is parallel to chord $\overline{ST}$. Chords $\overline{AS}$ and $\overline{WT}$ intersect at point B. If $m\overset{\frown}{WA} : m\overset{\frown}{AT} : m\overset{\frown}{ST} = 1:3:5$, find
a. $m\angle TBS$
b. $m\angle TWP$
c. $m\angle WPT$
d. $m\angle ASP$



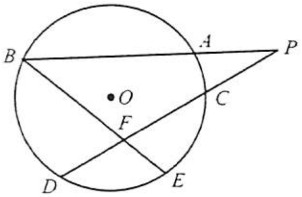
Exercise 23



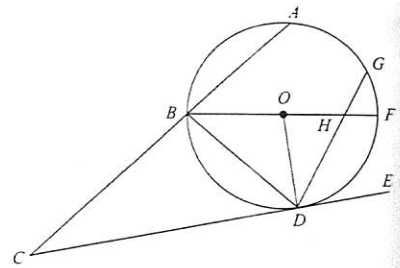
Exercise 24

24. In the accompanying diagram, $\overline{PCD}$ and $\overline{PBA}$ are secants from external point P to circle O. Chords $\overline{DA}$, $\overline{DEB}$, $\overline{CEA}$, and $\overline{CB}$ are drawn. $\overline{AB} \cong \overline{DC}$, $m\overset{\frown}{BC}$ is twice $m\overset{\frown}{AB}$, and $m\overset{\frown}{AD}$ is 60 more than $m\overset{\frown}{BC}$. Find:
a. $m\angle P$
b. $m\angle PCB$

25. In the accompanying diagram of circle O, $\overline{AB} \cong \overline{CD}$, $m\angle DFB = 70$, $m\overset{\frown}{AB} = 115$, and $m\overset{\frown}{DE} = 65$. Find the measure of $\angle P$.



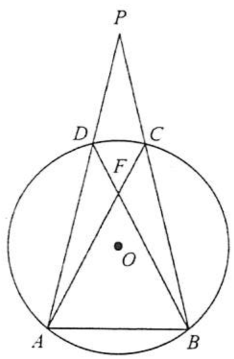
Exercise 25



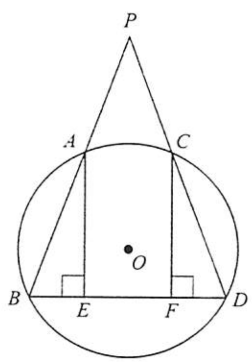
Exercise 26

26. In the accompanying diagram, circle O has radius $\overline{OD}$, diameter $\overline{BOHF}$, secant $\overline{CBA}$, and chords $\overline{DHG}$ and $\overline{BD}$; $\overline{CE}$ is tangent to circle O at D; $m\overset{\frown}{DF} = 80$; and $m\overset{\frown}{BA} : m\overset{\frown}{AG} : m\overset{\frown}{GF} = 3 : 2 : 1$.
Find $m\overset{\frown}{GF}$, $m\angle BHD$, $m\angle BDG$, $m\angle GDE$, $m\angle C$, and $m\angle BOD$.

27. Given: Secants $\overline{PDA}$ and $\overline{PCB}$ are drawn to circle O, $\overline{PDA} \cong \overline{PCB}$, chords $\overline{AC}$ and $\overline{BD}$ intersect at F.
Prove: $\overline{AF} \cong \overline{BF}$.



Exercise 27



Exercise 28

28. Given: Secants $\overline{PAB}$ and $\overline{PCD}$ are drawn to circle O, $\overline{PAB} \cong \overline{PCD}$, chords $\overline{AE} \perp \overline{BD}$, and $\overline{CF} \perp \overline{BD}$.
Prove: $\overline{AE} \cong \overline{CF}$.

7.5 SIMILAR TRIANGLES AND CIRCLES

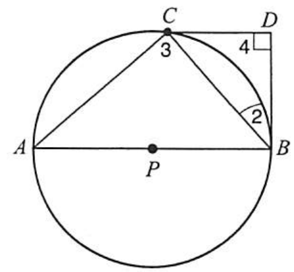
KEY IDEAS

Proving that triangles with chord, tangent, or secant segments as sides are similar can lead to some useful relationships involving the lengths of these segments.

Similar Triangle Proofs with Circles

Circle angle–measurement relationships can be used to help prove two triangles are similar.

Example 1



Given: $\overline{DB}$ is tangent to circle P at B, $\overline{AB}$ is a diameter, and $\overline{CD} \perp \overline{DB}$.

Prove: $\dfrac{AB}{BC} = \dfrac{BC}{CD}$.

Solution: See the proof.

PLAN: Prove that triangles ABC and CBD are similar.

Proof

- Angle 1 is an inscribed angle, and $\angle 2$ is formed by a tangent and a chord. Since the degree measure of each of these angles is one-half of the degree measure of the same arc, $\overset{\frown}{BC}$, $\angle 1 \cong \angle 2$.
- Angle 3 is inscribed in a semicircle, and $\angle 4$ is formed by perpendicular lines. Since each angle is a right angle, $\angle 3 \cong \angle 4$.
- Hence, $\triangle ABC \sim \triangle CBD$ by the AA Theorem of Similarity. Since the lengths of corresponding sides of similar triangles are in proportion:

$$\frac{AB}{BC} = \frac{BC}{CD}$$

Segments of Intersecting Chords

When two chords intersect inside a circle, similar triangles can be used to prove that the product of the lengths of the segments of one chord is equal to the product of the segments of the other chord. In Figure 7.22, $\triangle AED \sim \triangle CEB$ from which it follows:

$$AE \times EB = CE \times ED$$