17. On graph paper, draw the graph of the circle $x^2 + y^2 = 9$ and label it A.
   a. On the same set of axes, draw the image of circle A after the translation $(x, y) \rightarrow (x + 5, y - 3)$ and label it B.
   b. On the same set of axes, draw the image of circle B after a reflection in the x-axis and label it C. What is the area of the triangle formed by connecting the centers of circles A, B, and C?

18 a. On graph paper, draw the graph of circle O, which is represented by the equation $(x - 1)^2 + (y + 3)^2 = 16$.
   b. On the same set of axes, draw the image of circle O after the translation $(x, y) \rightarrow (x - 2, y + 4)$ and label it O'. Write an equation of circle O'.
   c. Write an equation of $\overleftrightarrow{OO'}$.

19–22. Solve each of the following line–circle systems of equations graphically.

19. $x^2 + y^2 = 25$
$x + 2y = 10$

20. $x^2 + (y - 2)^2 = 9$
$x - y = 1$

21. $(x - 2)^2 + (y - 3)^2 = 100$
$y + 13 = x$

22. $(x + 4)^2 + (y - 3)^2 = 25$
$2y + x = 7$

23–25. Solve each of the following line–parabola systems of equations graphically.

23. $y = x^2 + 4x - 1$
$y - x = 3$

24. $y = -x^2 - 2x + 8$
$y = x + 4$

25. $y = x^2 - 6x + 5$
$y + 7 = 2x$

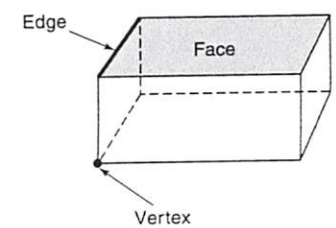
26. Graph on the same set of axes the two circles whose equations are $(x - 3)^2 + (y - 5)^2 = 25$ and $(x - 7)^2 + (y - 5)^2 = 9$. Write an equation of the line passing through the two points at which the circles intersect.

CHAPTER 10
AREA AND VOLUME OF SOLIDS

10.1 PRISMS AND CYLINDERS

KEY IDEAS

The solid figure counterpart of a polygon is called a *polyhedron.* A **polyhedron** is a solid figure whose sides or **faces** are polygons. The line segment where two faces intersect is called an **edge**. A **vertex** is a point at which three or more edges intersect.



Prisms

A **prism** is a polyhedron formed by connecting the corresponding vertices of two congruent polygons that lie in parallel planes. The two congruent polygons are called **bases.** Figure 10.1 shows a prism with pentagon bases. The lines joining the corresponding vertices of the bases are the **lateral edges** and are parallel to each other. The nonbase faces are called **lateral faces.**

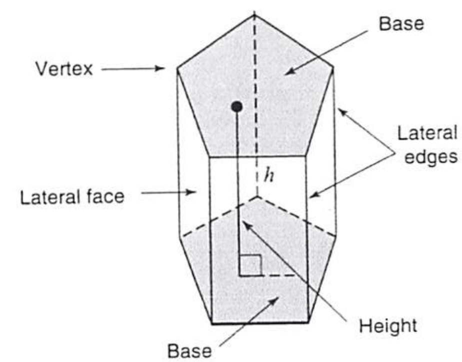
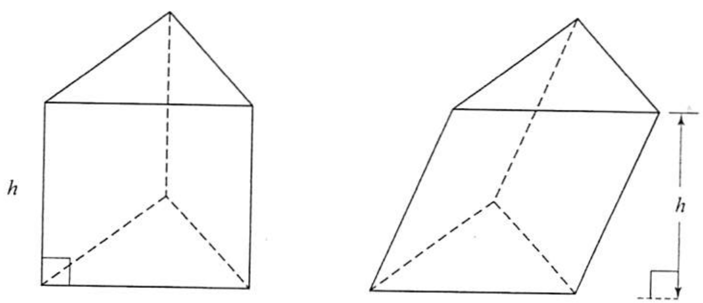


Figure 10.1 A prism with height *h* and congruent pentagon bases.

- Since the lateral edges are parallel to each other, the lateral faces of all prisms are parallelograms.
- An **altitude** of a prism is a line segment perpendicular to both bases and whose endpoints are in the planes of the bases. The length of the altitude is the **height** of the prism.
- A prism is named by the type of polygons that form their bases. A prism with triangles as bases, for example, is a *triangular prism*.

Right Prisms

A prism may be *right* or *oblique*, as in Figure 10.2, where h is the height of the prism. A **right prism** is one whose lateral edges are perpendicular to the bases. The bases of a right prism are aligned directly above the other. In a right prism, the length of a lateral edge represents the height of the prism, and the lateral faces are all rectangles.



Right triangular prism — Oblique triangular prism

Figure 10.2 Right and oblique prisms.

If the lateral edges of a prism are slanted because they are *not* perpendicular to a base, the prism is called an **oblique prism**. You can assume a prism is a right prism unless stated or drawn otherwise.

Volume of a Prism

The **volume** of a solid figure is the amount of space it occupies as measured by the number of nonoverlapping $1 \times 1 \times 1$ unit cubes that can exactly fill the interior of the solid.

Theorem: Volume of a Prism

- The volume, V, of a right rectangular prism (box) is the product of its three dimensions:

$$V_{\text{Box}} = \underbrace{\text{length} \times \text{width}}_{\text{Area of base}} \times \text{height} \quad \text{and} \quad V_{\text{Cube}} = (\text{edge})^3$$

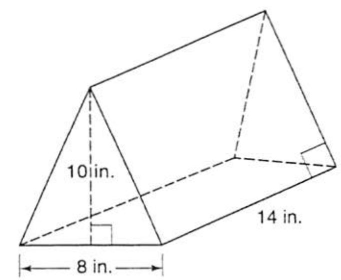
- The volume, V, of any prism is equal to the area of a base times the height:

$$V = B \times h$$

where B is the area of a base and h is the height. Thus, two prisms have equal volumes if their bases have equal areas and their heights are equal.

Example 1

Find the volume of the right triangular prism in the accompanying figure.



Solution:

- If B represents the area of a triangular base, then

$$B = \frac{1}{2} \times \text{base of triangle} \times \text{height of a triangle}$$
$$= \frac{1}{2} \times 8\text{ in} \times 10\text{ in}$$
$$= 40\text{ in}^2$$

- Since the height of the prism is the perpendicular distance between the two triangular bases, $h = 14$ in.
- Use the volume formula for a prism where $B = 40\text{ in}^2$ and $h = 14$ in:

$$V = B \times h$$
$$= 40\text{ in}^2 \times 14\text{ in}$$
$$= \mathbf{560\text{ in}^3}$$

Lateral Area of a Right Prism

The **lateral area** of a prism is the sum of the areas of only its nonbase (lateral) faces. To find the lateral area of the right prism in Figure 10.3, add the areas of its three lateral sides: $ABCD$, $ABEF$, and $ECDF$. Since $BC = AD = 12$ in and $AB = CD = 5$ in:

$$\begin{aligned} \text{Area rectangle } ABCD &= 5\times 12 = 60 \text{ in}^2\\ &+\\ \text{Area rectangle } ABEF &= 5\times 10 = 50 \text{ in}^2\\ &+\\ \text{Area rectangle } ECDF &= 5\times 10 = 50 \text{ in}^2\\ \hline \text{Lateral Area} &= 160 \text{ in}^2 \end{aligned}$$

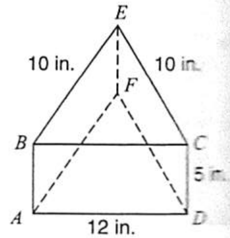


Figure 10.3 Right triangular prism

In a right prism, there is a relationship between the lateral area, height, and base perimeter:

$$\begin{aligned} \text{Lateral Area } &= 5\times 12 + 5\times 10 + 5\times 10\\ &= \underbrace{5}\times\underbrace{(12+10+10)}\\ &= \text{Height}\times\text{Perimeter of base} \end{aligned}$$

Theorem: Lateral Area of a Right Prism

If in a right prism, h is the height and p is the perimeter of a base, then

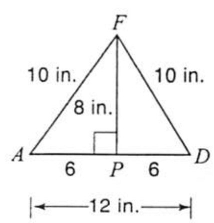
$$\text{Lateral Area (L.A.)} = h\times p.$$

The **surface area** (S.A.) of a prism is the total area of all of its faces including the two congruent bases.

Example 2

Find the volume of the prism in Figure 10.3.

Solution: The accompanying figure shows a two–dimensional view of the lower triangular base in which $\overline{FP}$ is the altitude drawn to the base of isosceles triangle AFD. The lengths of the sides of right triangle APF form a 6–**8**–10 Pythagorean triple in which $FP = 8$.



- Determine B, the area of the base of the prism:

$$\begin{aligned} \text{Area of } \triangle AFD &= \frac{1}{2}\times(AD)\times(FP)\\ &= \frac{1}{2}\times 12 \text{ in}\times 8 \text{ in}\\ &= 48 \text{ in}^2 \end{aligned}$$

Thus, $B = 48$ in^2.

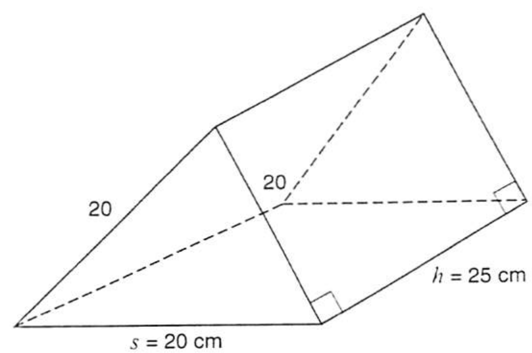
- Use the volume formula for a prism where $B = 48$ in^2 and $h = CD = 5$ in:

$$\begin{aligned} V &= B\times h\\ &= 48 \text{ in}^2\times 5 \text{ in}\\ &= \mathbf{240 \text{ in}^3} \end{aligned}$$

Example 3

In a right equilateral triangular prism, the length of a side of a base is 20 cm and a lateral edge measures 25 cm. Find the volume and lateral area of the prism.

Solution: In a right prism, the lateral edge represents an altitude, so $h = 25$ cm.



- Find the area of the equilateral triangle base using the formula $B = \frac{s^2}{4}\sqrt{3}$ where s is the length of a side of the triangle. Since it is given that $s = 20$ cm:

$$\begin{aligned} B &= \frac{(20)^2}{4}\sqrt{3}\\ &= \frac{400}{4}\sqrt{3}\\ &= 100\sqrt{3} \text{ cm}^2 \end{aligned}$$

- Use the volume formula for a prism where $B = 100\sqrt{3}$ and $h = 25$ cm:

$$\begin{aligned} V &= B \times h \\ &= (100\sqrt{3})(25) \\ &= \mathbf{2500\sqrt{3}\ cm^3.} \end{aligned}$$

- Use the formula for the lateral area of a prism where $h = 25$ cm and $p = 20 + 20 + 20 = 60$ cm:

$$\begin{aligned} \text{L.A.} &= h \times p \\ &= 25 \text{ cm} \times 60 \text{ cm} \\ &= \mathbf{1{,}500\ cm^2} \end{aligned}$$

Cylinders

A cylinder, like a prism, has two congruent bases in parallel planes. The bases of a cylinder, however, are circles rather than polygons. The radius, r, of a base is the **radius** of the cylinder. A cylinder may be right or oblique (slanted), as illustrated in Figure 10.4.

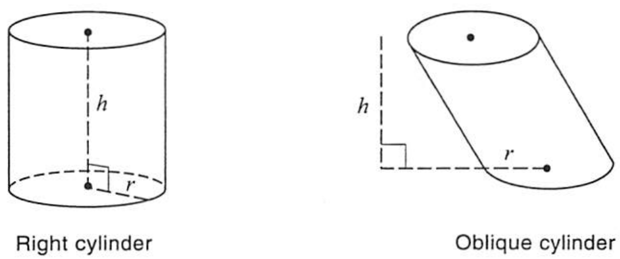


Figure 10.4 Right and oblique cylinders.

In a **right cylinder,** the line joining the centers of the two bases is perpendicular to the bases so that the centers of the bases are directly aligned. A cylinder is **oblique** if the line joining the centers is *not* perpendicular to the bases. The height, h, of a cylinder is the perpendicular distance between the two bases.

Volume and Area Formulas for a Cylinder

The volume and area formulas for a cylinder and right cylinder correspond to those of a prism and right prism except that for a cylinder πr^2 is substituted for the area of the base and $2\pi r$ is used as the perimeter (circumference) of the base.

> **Theorem: Cylinder Formulas**
> - The volume, V, of any cylinder is equal to the area of a base times the height:
>
> $$V = B \times h$$
>
> where $B = \pi r^2$ is the area of a circular base and h is the height.
> - If in a right cylinder, h is the height and p is the perimeter of a base, then
>
> $$\text{Lateral Area (L.A.)} = p \times h = 2\pi rh.$$

Example 4

A closed cylindrical can has a diameter of 1.0 ft and a height of 1.8 feet. Find
a. The volume of the can correct to the *nearest tenth* of a cubic foot.
b. The lateral area of the can.
c. The surface area of the can correct to the *nearest tenth* of a square foot.

Solution: Let $r = \frac{1}{2} \times 1.0$ ft $= 0.5$ ft and $h = 1.8$ feet.

a. Determine the volume of the cylinder using the formula $V = \pi r^2 h$:

$$\begin{aligned} V &= \pi \times (0.5)^2 \times 1.8 \\ &= \pi \times 0.25 \times 1.8 \\ &= 0.45\pi \text{ ft}^3 \end{aligned}$$

Multiply using the stored calculator value for π: $= 1.413716694 \text{ ft}^3$

Round: $= \mathbf{1.41\ ft^3}$

b. Use the formula for lateral area of a right cylinder:

$$\begin{aligned} \text{L.A.} &= 2\pi rh \\ &= 2\pi \times 0.5 \times 1.8 \\ &= \mathbf{1.8\pi\ ft^2} \end{aligned}$$

c. The surface area (S.A.) of the can is the sum of the areas of the two circular bases and the lateral area:

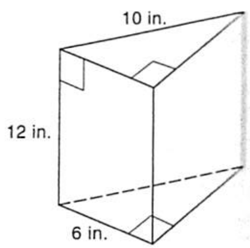
$$\begin{aligned} \text{S.A.} &= \text{L.A.} + 2\pi r^2 \\ &= 1.8\pi + 2\pi(0.5)^2 \\ &= 1.8\pi + 0.5\pi \\ &= 2.3\pi \text{ ft}^2 \\ &= \mathbf{7.2\ ft^2} \end{aligned}$$

Check Your Understanding of Section 10.1

A. Multiple Choice

1. What is the number of inches in the radius of a cylinder 2 inches in height if its volume is 288π in^3?
(1) 3 (2) 6 (3) 12 (4) 24

2. In the accompanying figure of a right prism, the bases are right triangles. Which statement about its volume (V) and lateral area (L.A.) is true?
(1) $V = 288$ in^3 and L.A. $= 288$ in^2
(2) $V = 288$ in^3 and L.A. $= 216$ in^2
(3) $V = 720$ in^3 and L.A. $= 288$ in^2
(4) $V = 720$ in^3 and L.A. $= 216$ in^2



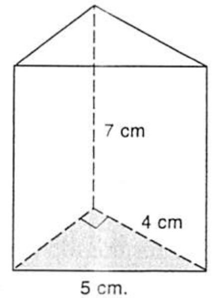
3. If in a right square prism, the length of each side of the two bases is doubled and the height is tripled, then the volume of the prism is
(1) multiplied by 12 (3) multiplied by 6
(2) multiplied by 8 (4) multiplied by 4

4. The amount of light produced by a cylindrical-shaped fluorescent light bulb depends on its lateral area. A certain cylindrical-shaped fluorescent light bulb is 36 inches in length, has a 1 inch diameter, and is manufactured to produce 0.283 watts of light per square inch. What is the best estimate for the total amount of light that it is able to produce?
(1) 32 watts (3) 48 watts
(2) 34 watts (4) 64 watts

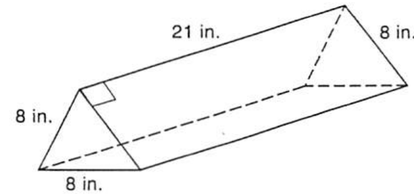
5. The bases of a right prism are right triangles whose legs measure 5 inches and 12 inches. If the lateral edge of the prism measures 15 inches, what is the number of square inches in the lateral area of the prism?
(1) 120 (2) 255 (3) 360 (4) 450

6. The lateral area of a right cylinder whose height is two times the length of the radius is 100π in^2. What is the number of inches in the height of the cylinder?
(1) 5 (2) 10 (3) 15 (4) 20

B. Show or explain how you arrived at your answer.

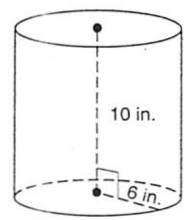


Exercise 7

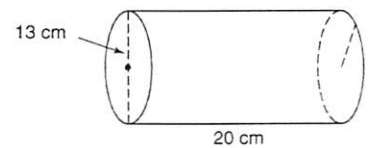


Exercise 8

7–8. For each right prism above, find the:
a. Lateral area.
b. Volume.



Exercise 9



Exercise 10

9–10. For each right cylinder above, find
a. The lateral area in terms of π.
b. The volume correct to the *nearest tenth* of a cubic unit.

11. The length of a side of the base of a right square prism is 12 cm. If the length of a lateral edge is 9 cm, find the surface area of the prism.

12. A prism has isosceles triangle bases whose legs measure 10 inches and base measures 16 inches. If the height of the prism is equal to the height of the triangular base, what is the volume of the prism?

13. A right circular cylinder has a lateral area of 306π ft^2. If the height of the cylinder is 17 ft, express in terms of π the number of cubic feet in the volume of the cylinder.

14. The base of a right prism is a right triangle whose legs measure 18 and 24 inches, respectively. If the length of a lateral edge is 16 inches, what is the surface area of the prism?

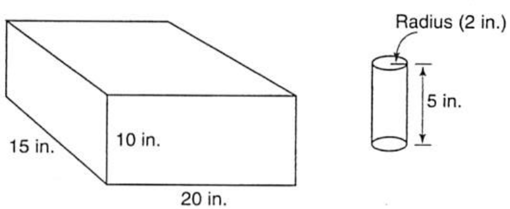
15. A piece of ice that has the shape of a right cylinder has a diameter of 6.0 cm and a height of 5.0 cm. If the ice cube melts at a constant rate of 13.0 cm^2 per minute, how many minutes elapse before the ice cube is completely melted? Round your answer to the *nearest hundredth* of a minute.

16. A cylindrical can is manufactured using aluminum for the top and bottom bases and tin for the remainder of the can. If the can is 10 cm in height and 6 cm in diameter, what percentage of the metal used to manufacture the can is tin? Round your answer to the *nearest tenth*.

17. The diameter of a cylindrical water storage tank is 12 feet and its height is 15 feet. A leak develops in the tank and water leaks out of it at a constant rate of 18.0 in^3/sec. If the tank is initially full, how many *hours* elapse before the tank is empty? Round off your answer to the *nearest hour*.

18. A hot water tank with a capacity of 85.0 gallons of water is being designed to have the shape of a right circular cylinder with a diameter [illegible] feet. Assuming that there are 7.48 gallons in 1 cubic foot, how high will the tank have to be? Round your answer to the *nearest tenth* of a foot.

19. A rectangle 6 inches in width and 8 inches in length is rotated in space about its longer side. Find in terms of π the volume and surface area of the solid generated.



20. In the accompanying diagram, a rectangular container with the dimensions 10 inches by 15 inches by 20 inches is to be filled with water, using a cylindrical cup whose radius is 2 inches and whose height is 5 inches. What is the maximum number of full cups of water that can be placed into the container without the water overflowing the container?

10.2 PYRAMIDS AND CONES

KEY IDEAS

Unlike prisms and cylinders, pyramids and cones have only one base.

Pyramids

A **pyramid** is a polyhedron formed by connecting all of the vertices of a polygon to a point in a different plane than the polygon, as in Figure 10.5. The polygon is called the **base**, and the point to which the vertices are connected is called the **vertex** of the pyramid. The line segments joining the vertices of the base to the vertex are the **lateral edges**. The lateral (nonbase) faces of a pyramid are always triangles. The **height**, h, of a pyramid is the perpendicular distance from the vertex to the plane of the base.

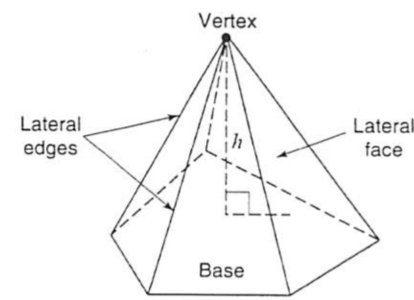


Figure 10.5 Pyramid with a pentagon base.

Volume of a Pyramid

Like a prism, the volume of a pyramid depends on the area of its base and the height.

Theorem: Volume of a Pyramid
The volume, V, of a pyramid is equal to one-third of the area of the base times the height:

$$V = \frac{1}{3}B \times h$$

where B is the area of the base and h is the height.

Regular Pyramid

The base of a pyramid can be any polygon. If the base is a regular polygon and its lateral edges are congruent, then the pyramid is a **regular pyramid**. Because the base of the pyramid in Figure 10.6 is a regular hexagon, the pyramid is classified as a regular hexagonal pyramid. You can assume a pyramid is regular unless stated or drawn otherwise.

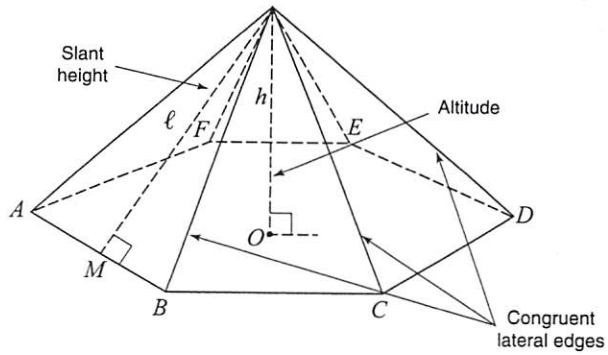


Figure 10.6 Regular hexagonal pyramid with center O, slant height ℓ ($\overline{VM}$), and $\overline{AM} \cong \overline{BM}$.

In Figure 10.6, VM represents the slant height. The **slant height** of a regular pyramid is the perpendicular distance from the vertex of the pyramid to a side of the base. The slant height is usually denoted by ℓ. Because the lateral faces of a regular pyramid are congruent isosceles triangles, the slant height is the perpendicular bisector of the side to which it is drawn. In any regular pyramid,

- The lateral faces are congruent isosceles triangles.
- The altitude drawn from the vertex intersects the base at its center.
- The altitude from the vertex and the slant height determine a right triangle, as illustrated in the regular square pyramid in Figure 10.7. Because O is the center of the square base and M is the midpoint of $\overline{AB}$, OM is one-half the length of a side of the square base.

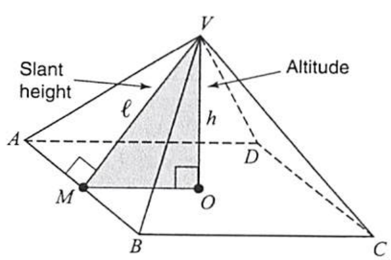


Figure 10.7 Right triangle determined by slant height $\overline{VM}$ and altitude $\overline{VO}$.

Lateral Area of a Regular Pyramid

The lateral area of a regular pyramid depends on its slant height and perimeter.

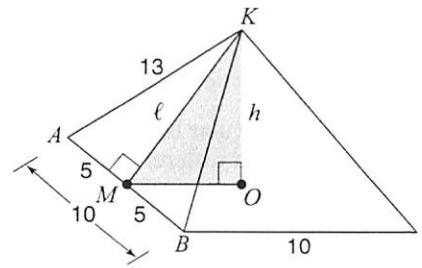
Theorem: Lateral Area Formula for a Regular Pyramid
If in a regular pyramid ℓ is the slant height and p is the perimeter of the base, then:

$$\text{L.A.} = \frac{1}{2}p\ell$$

Example 1

The perimeter of the base of a regular square pyramid is 40 cm and a lateral edge measures 13 cm. Find

a. The lateral area of the pyramid.
b. The height of the pyramid in radical form.
c. The volume of the pyramid correct to the *nearest tenth* of a cubic centimeter.



Solution:

a. First find the slant height.

- Each side of the base measures $\frac{40 \text{ cm}}{4} = 10$ cm. Because it is given that a lateral edge measures 13 cm, $KA = 13$ cm.
- Lateral face AKB is an isosceles triangle in which altitude $\overline{KM}$ bisects base $\overline{AB}$. Thus, $\triangle AMK$ is a 5–**12**–13 right triangle in which KM, the slant height, is 12 cm.
- Use the formula for lateral area of a pyramid where $p = 40$ cm and $\ell = 12$ cm:

$$\begin{aligned}\text{L.A.} &= \frac{1}{2}p\ell \\ &= \frac{1}{2}(40 \text{ cm})(12 \text{ cm}) \\ &= \mathbf{240\ cm^2}\end{aligned}$$

b. To find the height, h, of the pyramid, apply the Pythagorean Theorem in right triangle KOM where the length of hypotenuse $\overline{KM}$ is 12. Since $\overline{OM}$ is one-half of the side length of the square, $OM = \frac{1}{2} \times 10 = 5$. Thus,

$$\begin{aligned} h^2 + (OM)^2 &= (KM)^2 \\ h^2 + 5^2 &= 12^2 \\ h^2 &= 144 - 25 \\ \mathbf{h} &= \mathbf{\sqrt{119}\ cm} \end{aligned}$$

c. Use the formula for the volume of a pyramid where $B = 10\text{ cm} \times 10\text{ cm} = 100\text{ cm}^2$ and $h = \sqrt{119}$ cm:

$$\begin{aligned} V &= \frac{1}{3}Bh \\ &= \frac{1}{3} \times 100\text{ cm}^2 \times \sqrt{119}\text{ cm} \end{aligned}$$

Use your calculator: $= \mathbf{363.6\ cm^3}$

Cones

A **cone**, unlike a pyramid, has a circular base with each point on its circumference joined to a single point in a different plane called the **vertex** of the cone. A cone may be right or oblique, as in Figure 10.8.

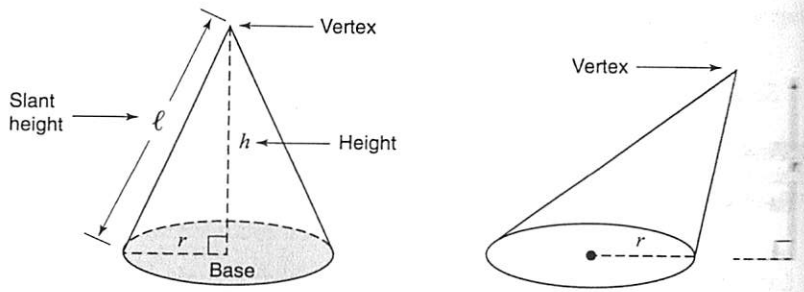


Figure 10.8 Right and oblique cones.

A **right cone** is a cone in which the line segment joining the vertex and the center of the base is perpendicular to the plane of the base, as in Figure 10.8. You can assume a cone is a right circular cone unless stated or drawn otherwise. The **slant height**, ℓ, of a right circular cone is the distance from the vertex to any point on the circumference of the circular base. The volume and area formulas for a cone correspond to those of a pyramid except that πr^2 and $2\pi r$ are substituted for the area and the perimeter of the base, respectively.

Theorem: Cone Formulas

- The volume, V, of any cone is equal to one-third of the area of the base times the height:

$$V = \frac{1}{3}B \times h = \frac{1}{3}\pi r^2 h$$

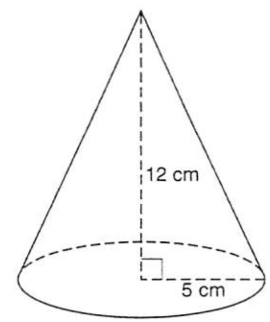
where B is the area of the base, h is the height, and r is the radius of the base.

- If in a right circular cone ℓ is the slant height and r is the radius, then

$$\text{L.A.} = \frac{1}{2} \times \underbrace{\text{Circumference}}_{\text{"base perimeter"}} \times \ell = \frac{1}{2}(2\pi r)\ell = \pi r\ell$$

Example 2

Find the volume and surface area of the cone in the accompanying figure. Answers may be left in terms of π.



Solution: To find the volume of the cone, use the volume formula where $r = 5$ cm and $h = 12$ cm:

$$\begin{aligned} V &= \frac{1}{3}\pi r^2 h \\ &= \frac{1}{3}\pi \times 5^2 \times 12 \\ &= \mathbf{100\ \pi\ cm^3} \end{aligned}$$

To find the surface area, first find the lateral area.

- Since the slant height is the hypotenuse of a 5–12–*13* right triangle, $\ell = 13$ cm. Hence,

$$\text{L.A.} = \pi r\ell = \pi \times 5\text{ cm} \times 13\text{ cm} = 65\pi\text{ cm}^2.$$

- The surface area includes both the lateral area and the area of the base:

$$\text{S.A.} = \text{L.A.} + \pi r^2 = 65\pi\mathbf{cm^2} + \pi(5\text{ cm})^2 = \mathbf{90\pi cm^2}.$$

Table 10.1 summarizes volume and lateral area formulas where B = area of base, h = height, p = base perimeter, r = base radius, and ℓ = slant height. The formulas for lateral area only apply to right prisms, right cylinders, regular pyramids, or right cones.

Table 10.1 Formulas for Volume and Lateral Area.

Prism	Cylinder	Pyramid	Cone
• $V = Bh$	• $V = Bh$ where $B = \pi r^2$	• $V = \frac{1}{3}Bh$	• $V = \frac{1}{3}Bh$ where $B = \pi r^2$
• L.A. $= hp$	• L.A. $= 2\pi rh$	• L.A. $= \frac{1}{2}p\ell$	• L.A. $= \pi r\ell$

Check Your Understanding of Section 10.2

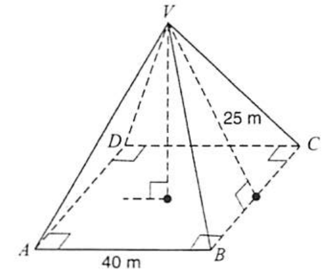
A. Multiple Choice

1. For any regular pyramid with height h and slant height ℓ, which statement is always true?
(1) $h > \ell$
(2) $h < \ell$
(3) $h = \ell$
(4) $\ell >$ lateral edge

2. If a regular square pyramid with side length s and a right cone with radius r have congruent altitudes and equal volumes, then
(1) $s = \sqrt{\pi r}$
(2) $s = \frac{\sqrt{r}}{\pi}$
(3) $s = \pi\sqrt{r}$
(4) $s = r\sqrt{\pi}$

3. For any right cone, which statement is always true?
(1) lateral area > area of base
(2) area of base > lateral area
(3) area of base = lateral area
(4) surface area = 2 × area of base

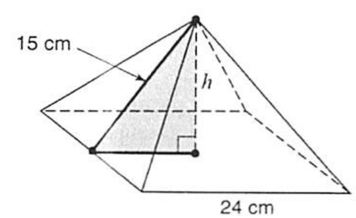
4. In the accompanying figure of a regular square pyramid, the length of a side of the base is 40 meters and the slant height measures 25 meters. Which statement about its volume (V) and lateral area (L.A.) is true?
(1) $V = 8{,}000$ m^3 and L.A. $= 2{,}000$ m^2
(2) $V = 8{,}000$ m^3 and L.A. $= 1{,}200$ m^2
(3) $V = 2{,}000$ m^3 and L.A. $= 2{,}000$ m^2
(4) $V = 2{,}000$ m^3 and L.A. $= 1{,}200$ m^2



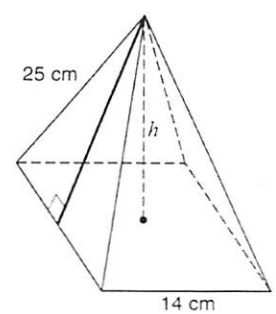
5. A right cone has a diameter of 18 feet and a slant height of 15 feet. Which statement about its volume (V) and lateral area (L.A.) is true?
(1) $V = 405\pi$ ft^3 and L.A. $= 135\pi$ ft^2
(2) $V = 405\pi$ ft^3 and L.A. $= 324\pi$ ft^2
(3) $V = 324\pi$ ft^3 and L.A. $= 135\pi$ ft^2
(4) $V = 324\pi$ ft^3 and L.A. $= 324\pi$ ft^2

B. Show or explain how you arrived at your answer.

6. In the accompanying diagram of a square pyramid, the slant height is 15 cm, and the length of a side of the base is 24 cm. Find the lateral area and volume of the prism.



Exercise 6

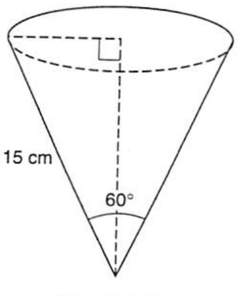


Exercise 7

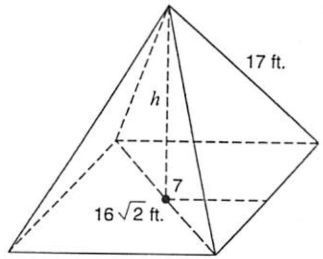
7. In the accompanying diagram of a square pyramid, the length of a lateral edge is 25 cm and the length of a side of the base is 14 cm. Find
a. The surface area of the pyramid
b. The volume of the pyramid correct to the *nearest tenth* of a cubic centimeter

8. A right circular cone has a lateral surface area of 80π square inches. If the slant height of the cone is 10 inches, what is the volume of the cone?

9. In the accompanying figure of a right cone, the vertex angle measures 60° and the slant height is 15 cm. Find, to the *nearest tenth* of a cubic centimeter, the volume of the cone.



Exercise 9



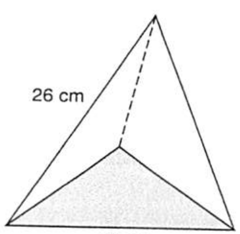
Exercise 10

10. In the accompanying figure of a regular pyramid, the length of a diagonal of the square base is $16\sqrt{2}$ ft. If a lateral edge measures 17 ft, find the number of cubic feet in the volume of the pyramid. Round your answer to the *nearest tenth* of a cubic foot.

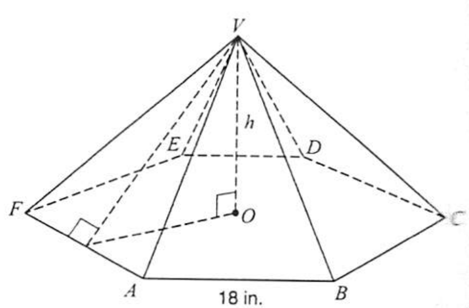
11. The altitude of a right circular cone makes an angle of 45° with the slant height. If the radius of the base is 8 inches, what is the volume of the cone in terms of π?

12. The lateral area of a regular square pyramid is 1,040 ft^2. If the slant height is 26 ft, what is the number of cubic feet in the volume of the pyramid?

13. In the accompanying diagram of a regular triangular pyramid, the length of each side of the base is 20 cm and the length of a lateral edge is 26 cm. Find
a. The lateral area of the pyramid
b. The volume of the pyramid to the *nearest cubic centimeter*



Exercise 13

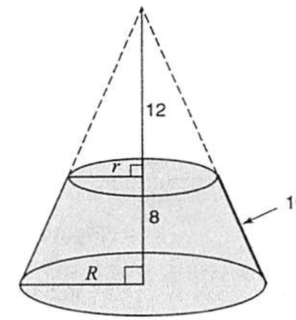


Exercises 14–15

14–15. In the accompanying diagram of a regular hexagonal pyramid, the length of each side of the base is 18 cm, and the length of a lateral edge is 41 cm.

14. Find the lateral area of the pyramid.

15. Find the volume of the pyramid correct to the *nearest cubic centimeter*.



16. A lamp shade with a circular base has the shape of a solid called a *frustrum*. If a plane parallel to the base of a cone intersects the cone below its vertex, the truncated part of the cone between the slicing plane and the base is called the **frustrum** of the cone. In the accompanying figure, the shaded region represents a frustrum of a right cone in which the portion of the original cone that lies 12 inches below its vertex has been cut off. Find
a. The radius, R, of the original cone
b. The volume of the frustrum expressed in terms of π

17. A cone has a volume of 135 in^3. A plane parallel to the base divides the cone into two parts such that one part is a cone whose height is two-thirds that of the height of the original cone. Find the volumes of this smaller cone and the remaining frustrum.

10.3 SPHERES AND SIMILAR SOLIDS

KEY IDEAS

The solid figure counterpart of a circle is a *sphere*. Two solids are **similar** if they have the same shape and their corresponding dimensions are in proportion. The ratio of corresponding linear dimensions of two similar solids is called the **similarity ratio**. Any pair of cubes or pair of spheres are similar.

Spheres and Great Circles

A **sphere** is the set of all points in space at a fixed distance from a given point called the **center**. The **radius** of a sphere is the length of a line segment joining the center of the sphere to any point on the sphere. See Figure 10.9.

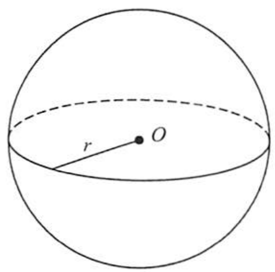


Figure 10.9 Sphere with center O and radius r.

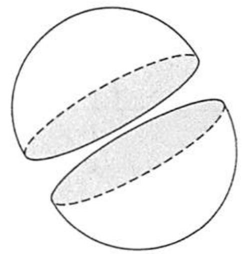


Figure 10.10 A great circle divides the sphere into two hemispheres.

A **great circle** is the largest circle that can be drawn on a sphere. It has the same center as the sphere and divides the sphere into two equal parts called **hemispheres**, as in Figure 10.10.

- The intersection of a plane and a sphere is a circle. A great circle is the intersection of a plane passing through the center of a sphere. The circumference of a great circle may be drawn through any two points on the surface of a sphere as these two points and the center of the sphere determine a plane. See Figure 10.11.

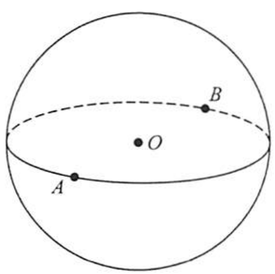


Figure 10.11 Drawing a great circle given two points on the surface of the sphere.

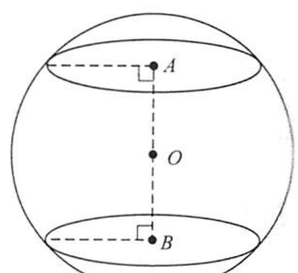


Figure 10.12 Planes intersecting sphere O in circles A and B.

- Two planes equidistant from the center of the sphere and intersecting the sphere do so in congruent circles. If in sphere O shown in Figure 10.12, $OA = OB$, then circles A and B are congruent.
- The surface area and volume of a sphere depends on its radius.

Theorem: Sphere Formulas

If a sphere has radius r, then

$$\text{Surface Area} = 4\pi r^2 \quad \text{and} \quad \text{Volume} = \frac{4}{3}\pi r^3$$

Example 1

Find the volume of a sphere whose surface area is 144π in^2.

Solution: Use the formula for surface area to find the radius of the sphere:

$$\text{S.A.} = 4\pi r^2 = 144\pi \text{ in}^2$$
$$r^2 = \frac{144\pi}{4\pi} = 36 \text{ in}^2$$
$$r = 6 \text{ in}$$

Now use the formula for volume of a sphere where $r = 6$ in

$$V = \frac{4}{3}\pi r^3$$
$$= \frac{4}{3}\pi(6^3)$$
$$= \frac{4\pi(216)}{3}$$
$$= 288\pi$$

The volume of the sphere is **288π in^3**.

Comparing Areas and Volumes of Similar Solids

If the ratio of corresponding linear dimensions of two similar solids is $\frac{a}{b}$, then

- The ratio of their corresponding areas is $\left(\frac{a}{b}\right)^2$.
- The ratio of their corresponding volumes is $\left(\frac{a}{b}\right)^3$.

For example, if the ratio of the heights of two similar cylinders is $\frac{2}{3}$, then the ratio of their lateral areas is $\left(\frac{2}{3}\right)^2$ or $\frac{4}{9}$, and the ratio of their volumes is $\left(\frac{2}{3}\right)^3$ or $\frac{8}{27}$.

Example 2

Two spheres have radii of 8 cm and 20 cm. If the volume of the smaller sphere is 104π cm^3, what is the volume of the larger sphere?

Solution: Since the similarity ratio is $\frac{8}{20}$ or, equivalently, $\frac{2}{5}$:

$$\frac{\text{Volume of smaller sphere}}{\text{Volume of larger sphere}} = \left(\frac{2}{5}\right)^3$$

If x represents the volume of the larger sphere, then

$$\frac{104\pi}{x} = \frac{8}{125}$$

$$8x = (104\pi)(125)$$

$$\frac{8x}{8} = \frac{13{,}000\pi}{8}$$

$$x = 1{,}625\pi$$

The volume of the larger sphere is **$1{,}625\pi$ cm^3**

Example 3

The lateral areas of two similar prisms are 324 in^2 and 729 in^2. If the volume of the larger prism is 2,916 in^3, what is the volume of the smaller prism?

Solution: First find the similarity ratio. The lateral areas of the two similar prisms have the same ratio as the *square* of the similarity ratio. If $\frac{a}{b}$ represents the similarity ratio, then

$$\left(\frac{a}{b}\right)^2 = \frac{324}{729}$$

$$\frac{a}{b} = \frac{\sqrt{324}}{\sqrt{729}}$$

$$= \frac{18}{27}$$

$$= \frac{2}{3}$$

Let x represent the volume of the smaller prism. Since the volumes of the two similar prisms have the same ratio as the *cube* of the similarity ratio:

$$\frac{\text{Volume of smaller prism}}{\text{Volume of larger prism}} = \left(\frac{2}{3}\right)^3$$

$$\frac{x}{2916} = \frac{8}{27}$$

$$27x = 23{,}328$$

$$\frac{27x}{27} = \frac{23{,}328}{27}$$

$$x = 864$$

The volume of the smaller prism is **864 in^3**.

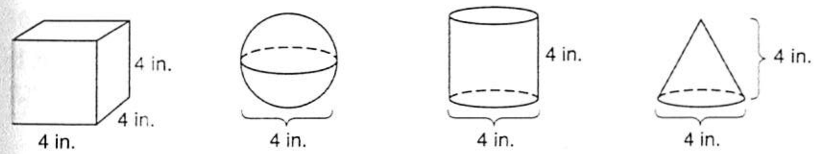
Check Your Understanding of Section 10.3

A. *Multiple Choice*

1. If the diameter of a sphere is tripled, the volume of the sphere is multiplied by
(1) 4 (2) 9 (3) 16 (4) 27

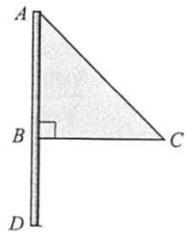
2. The density of lead is approximately $0.41\ \frac{\text{pounds}}{\text{in}^3}$. What is the approximate weight, in pounds, of a lead ball that has a 5 inch diameter?
(1) 26.8 (2) 78.5 (3) 80.4 (4) 214.7

3. Which diagram represents the figure with the greatest volume?



(1) (2) (3) (4)

4. Triangle $\overline{ABC}$ represents a metal flag on pole $\overline{ABD}$, as shown in the accompanying diagram. On a windy day the triangle spins around the pole so fast that it looks like a three-dimensional shape. Which shape would the spinning flag create?
   (1) sphere
   (2) cone
   (3) right cylinder
   (4) pyramid



5. What is the best approximation of the number of square inches in the surface area of a spherical ball that fits tightly in a cube-shaped box with edges 8 inches in length?
   (1) 67 (2) 201 (3) 268 (4) 804

6. If the volume of a sphere is $7{,}776\pi$ cm^3, what is the number of square centimeters in the surface area of the sphere?
   (1) 324π (2) 648π (3) $1{,}296\pi$ (4) $2{,}592\pi$

7. If the ratio of the volumes of two similar solids is 27 to 64, what is the ratio of their areas?
   (1) $\sqrt{27}$ to 8 (2) 3 to 8 (3) 9 to 16 (4) 3 to 4

8. The surface areas of two similar prisms are 112 in^2 and 63 in^2. The ratio of the volume of the larger prism to the volume of the smaller prism is
   (1) 4 to 3
   (2) 16 to 9
   (3) 64 to 27
   (4) 256 to 81

B. *Show or explain how you arrived at your answer.*

9. Tamika has a hard rubber ball whose circumference measures 13 inches. She wants to box it for a gift but can only find cube-shaped boxes of sides of 3 inches, 4 inches, 5 inches, or 6 inches.
   a. What is the *smallest* box that the ball will fit into with the top on?
   b. Tamika has a square sheet of wrapping paper that measures 1 ft by 1 ft. Assuming no waste, does Tamika have enough wrapping paper to completely cover the box? Explain your answer.

10. A solid sphere with a radius of 8 inches weighs 32 pounds. What is the weight, in pounds, of a sphere made of the same material whose radius is 6 inches?

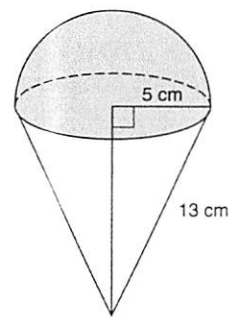
11. A ball is placed inside a cube-shaped boxed that measures 6 inches on each side. When the box is closed, each of its sides touches the ball. What is the approximate volume of the enclosed space between the sphere and the sides of the box correct *to the nearest cubic inch*?

12. A bookend is shaped like a pyramid and weighs 0.24 pounds. How many pounds does a similarly shaped bookend weigh if it is made of the same material and each corresponding dimension is $2\frac{1}{2}$ times as large?

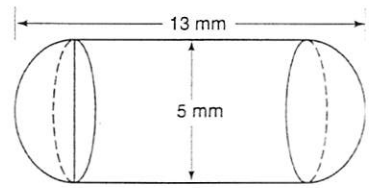
13. The lateral areas of two similar cylindrical cans are 567 in^2 and 1,008 in^2. If the volume of the smaller can is 1,161 in^3, what is the volume of the larger can?

14. The volumes of two similar pyramids are 2,744 cm^3 and 9,261 cm^3. If the lateral area of the smaller pyramid is 1,624 cm^2, what is the lateral area of the larger pyramid?

15. In the accompanying figure, a cone with a solid hemisphere at its top has a slant height of 13 cm. If the radii of the cone and hemisphere are 5 cm, find the volume of the figure correct to the *nearest cubic centimeter*.



Exercise 15



Exercise 16

16. In the accompanying figure, a vitamin tablet is 13 mm long and 5 mm in diameter. Find the volume of the tablet to the *nearest cubic millimeter*.

17. A sealed cylindrical can holds three tennis balls each with a diameter of 2.5 inches. If the can is designed to have the smallest possible volume, find the number of cubic inches of unoccupied space inside the can correct to the *nearest tenth of a cubic inch*.